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Applications on Discrete Laplace with Riemannian Geometric

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ABSTRACT :

In this paper uniform contractible Laplace on Euclidean space is manifolds ,we also construct a pair uniformly contractible Laplace on operators Riemannian metrics on R^n so that the resulting manifolds M and M' are bounded is close to a homeomorphism and a proof the Laplace operator on compact Riemannian manifolds.

Key words: Basic on Laplace operator – vector and convector on manifolds fields-spectrum the Laplace in Riemannian manifolds- Deferential Laplace on p-forms – Intrinsic on Bounded manifolds .

INTRODUCTION

(Laplace – Beltrami operator plays a fundamental role in Riemannian geometric .In real applications, smooth metric surface is usually represented as triangulated mesh the manifold heat kernel is estimated from the discrete Laplace operator- Discrete Laplace – Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications including mesh parameterization segmentation. The Riemannian manifold with boundary, in the Euclidean domain the interior geometry is given ,flat and trivial, and the interesting phenomena come from the shape of the boundary ,Riemannian manifolds have no boundary, and the geometric phenomena are those of the interior . The present paper is an introduction, so we have to refrain from saying too must . To any compact Riemannian manifold (M,g) is boundary we associate second- order (P.D.E) , the Laplace operator Δ is defined by $\Delta(f)=-div(grad\, f)$ for $f\in L^2(M,g)$. We also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with the set of eigenvalues of Δ is called the spectrum of Δ . The manifolds to investigated which are manifolds of systems of differential polynomials in a single unknown , possess a degree of analogy to bounded sets of numbers . They are manifolds which may be said (not to contain infinity as a solution) U_α where each set U_α is homeoomorphic, via some homeomorphism h_α to an open subset of Euclidean space R^n , valued function converse a vector valued function given curve , the tangent line at the point.

II. BASIC ON LAPLCES OPERATORES

2.1 Basic on Laplacian Manifolds

Definition 2.1.1

A topological manifold M of dimension n, is a topological space with the following properties:

(i) M is a Hausdorff space . For ever pair of points $p, g \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $g \in V$.

(ii) M is second countable . There exists accountable basis for the topology of M .

(iii) M is locally Euclidean of dimension n . Every point of M has a neighborhood that is homeomorphic to an open subset of R^n .

Definition 2.1.2

A coordinate chart or just a chart on a topological $n-$manifold M is a pair (U,φ), Where U is an open subset of M and $\varphi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi\ (U) \subset R^n$.

Examples 2.1.3

Let S^n denote the (unit) $n-$sphere, which is the set of unit vectors in R^{n+1}: $S^n = \{x \in R^{n+1} : |\ x\ | = 1\}$ with the subspace topology, S^n is a topological $n-$manifold.

Definition 2.1.4

The $n-$dimensional real (complex) projective space, denoted by $P_n(R)$ or $P_n(C)$), is defined as the set of 1-dimensional linear subspace of R^{n+1} $or\, C^{n+1}$), $P_n(R)$ or $P_n(C)$ is a topological manifold.

Definition 2.1.5

For any positive integer n, the $n-$torus is the product space $T^n = (S^1 \times ... \times S^1)$ it is an $n-$dimensional topological manifold (The 2-torus is usually called simply the torus).

Definition2.1.6

The boundary of a line segment is the two end points; the boundary of a disc is a circle. In general the boundary of an $n-$manifold is a manifold of dimension $(n-1)$, we denote the boundary of a manifold M as ∂M . The boundary of boundary is always empty, $\partial\partial M = \phi$

Lemma 2.1.7

Every topological manifold has a countable basis of Compact coordinate balls. Every topological manifold is locally compact.

Definitions 2.1.8

Let M be a topological space n-manifold. If $(U,\varphi),(V,\psi)$ are two charts such that $U \cap V \neq \phi$, the composite map

(1) $$\left\{\ \psi \circ \varphi^{-1} : \varphi\ (U \cap V) \to \psi\ \ (U \cap V)\ \right\}$$

is called the transition map from φ to ψ .

Definition 2.1.9
An atlas A is called a smooth atlas if any two charts in A are smoothly compatible with each other. A smooth atlas A on a topological manifold M is maximal if it is not contained in any strictly larger smooth atlas. (This just means that any chart that is smoothly compatible with every chart in A is already in A) .

Definition 2.1.10
A smooth structure on a topological manifold M is maximal smooth atlas. (Smooth structure are also called differentiable structure
or C^∞ structure by some authors).

Definition 2.1.11
A smooth manifold is a pair M where M is a topological manifold and A is smooth structure on M . When the smooth structure is understood, we omit mention of it and just say M is a smooth manifold.

Definition 2.1.12
Let M be a topological manifold :
(i) Every smooth atlases for M is contained in a unique maximal smooth atlas.
(ii) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is smooth atlas.

Definition 2.1.13
Every smooth manifold has a countable basis of pre-compact smooth coordinate balls. For example the General Linear Group The general linear group $GL(n,R)$ is the set of invertible $n\times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2- dimensional vector space $M(n,R)$, namely the set where the (continuous) determinant function is nonzero.

Definition 2.1.14
Let M be a smooth manifold and let p be a point of M . A linear map $X : C^\infty(M)\to R$ is called a derivation at p if it satisfies :
(2) $$X(fg) = f(p)Xg + g(p)Xf$$
for all $f,g\in C^\infty(M)$. The set of all derivation of $C^\infty(M)$ at p is vector space called the tangent space to M at p , and is denoted by [T_pM]. An element of T_pM is called a tangent vector at p .

Lemma 2.1.15
Let M be a smooth manifold, and suppose $p\in M$ and $X\in T_pM$. If f is a const and function, then $Xf = 0$. If $f(p) = g(p) = 0$, then $X(fp) = 0$.

Definition2.1.16
If γ is a smooth curve (a continuous map $\gamma : J\to M$,where $J\subset R$ is an interval) in a smooth manifold M , we define the tangent vector to γ at $t_\circ\in J$ to be the vector $\gamma'(t_\circ) = \gamma_*\left(\frac{d}{dt}|_{t_\circ}\right)\in T_{\gamma(t_\circ)}M$, where $d/dt|_{t_\circ}$ is the standard coordinate basis for $T_{t_\circ}R$. Other common

notations for the tangent vector to γ are $\left[\gamma^*(t_\circ), \frac{d\gamma}{dt}(t_\circ)\right]$ and $\left[\frac{d\gamma}{dt}|_{t=t_\circ}\right]$. This tangent vector acts on functions by :

(3) $$\left\{\gamma'(t_\circ)\ f = \left(\ \gamma_*\frac{d}{dt}|_{t_\circ}\right)\ f = \frac{d}{dt}|_{t_\circ}\left(f\circ\gamma\right) = \frac{d(f\circ\gamma)}{dt}(t_\circ)\right\}$$

Lemma 2.1.17
Let M be a smooth manifold and $p \in M$. Every $X \in \left(T_p M\right)$ is the tangent vector to some smooth curve in M .

Definition 2.1.18
A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m: G\times G \to G$ and inversion map $m: G \to G$, given by $m(g,h) = gh,\ \ i(g) = g^{-1}$, are both smooth. If G is a smooth manifold with group structure such that the map $G\times G \to G$ given by $(g,h) \to gh^{-1}$ is smooth, then G is a Lie group. Each of the following manifolds is a lie group with indicated group operation . The general linear group $GL(n,R)$ is the set of invertible $n\times n$ matrices with real entries. It is a group under matrix multiplication, and it is an open sub-manifold of the vector space $M(n,R)$, multiplication is smooth because the matrix entries of A and B. Inversion is smooth because Cramer's rule expresses the entries of A^{-1} as rational functions of the entries of A. The $n-$torus $T^n = (S^1 \times ... \times S^1)$ is an $n-$dimensional a Belgian group.

Definition 2.1.19
Let V and W be smooth vector fields on a smooth manifold M. Given a smooth function $f: M \to R$, we can apply V to f and obtain another smooth function Vf , and we can apply W to this function, and obtain yet another smooth function $\left(WV\right)f = W(Vf)$. The operation $f \to WV\ f$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following for example shows let $V = \left(\frac{\partial}{\partial x}\right)$ and $W = \left(\frac{\partial}{\partial y}\right)$ on R^n , and let $f(x, y) = x, g(x, y) = y$.Then direct computation shows that $VW(f\ g) = 1$, while $\left(f\,V\,W\ g + g\ V\ W\ f\right) = 0$, so VW is not a derivation of $C^\infty(R^2)$. We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function $WV\ f$. Applying both of this operators to f and subtraction, we obtain an operator $[V,W]: C^\infty(M) \to C^\infty(M)$, called the Lie bracket of V and W, defined by $[V,W]\ f\ =\ \left(VW\right)f - \left(WV\right)f$. This operation is a vector field. The Smooth of vector Field is Lie bracket of any pair of smooth vector fields is a smooth vector field.

Lemma 2.1.20
The Lie bracket satisfies the following identities for all $V,W,X \in\ (M)$. Bilinearity: $\forall a,b \in R$,

(4) $$\left\{\begin{array}{l}[\ aV + bW, X\ ] = a\ [V, X] + b\ [W, X]\\ [\ X, aV + bW\ ] = a\ [X,V] + b\ [X,W]\end{array}\right\}$$

(i) Ant symmetry $[V,W] = -\ [W,V]$.

(ii) Jacobi identity $[V,[\,W,X\,]] + [\,W,[\,X,V]\,] + [\,X,[V,W]\,] = 0$. For $f, g \in C^\infty(M)$

(5) $$[\ f\ V, g\ W\ ] = f\ g\ [\ V,W\ ] + (\ f\,V\,g)\ W - (g\ W\ f)\ V$$

2.3 Convector Fields

Let V be a finite n– dimensional vector space over R and let V^* denote its dual space. Then V^* is the space whose elements are linear functions from V to R , we shall call them Convectors. If $\sigma \in V^*$ then $\sigma : V \to R$ for the any $v \in V$, we denote the value of σ on v by $\sigma(v)$ or by $\langle v, \sigma \rangle$. Addition and multiplication by scalar in V^* are defined by the equations:

(6) $$\{ \ (\sigma_1 + \sigma_2)\ (v) = \sigma_1(v) + \sigma_2(v)\ ,\ \ (\alpha\sigma)\ (v) = \alpha\ (\sigma\ (v)\ )\ \}$$

Where $,\sigma, \alpha\sigma \in V^*$ and $\alpha \in R$.

Proposition 2.3.1

Let V be a finite- dimensional vector space. If $(E_1,...,E_n)$ is any basis for V ,then the convectors $(\omega^1,...,\omega^n)$ defined by:

(7) $$\omega^i(E_j) = \delta^i_j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

form a basis for V^*,called the dual basis to (E_j) .Therefore, $\dim V^* = \dim V$.

Definition 2.3.2

AC^r – Convector field σ on M , $r \geq 0$, is a function which assigns to each $\beta \in M$ a convector $\sigma_p \in T^*_P(M)$ in such a manner that for any coordinate neighborhood U, ϕ with coordinate frames $E_1,..,E_n$, the functions $\sigma(E_i)$, $i = 1,.....,n,$ are of class C^r on U . For convenience, (Convector field) will mean C^∞ – convector field.

Remark 2.3.3

It is important to note that a C^r – Convector field σ defines a map $\sigma : \mathcal{X}(\mathrm{M}) \to C^r(M)$, which is not only R – Linear but even $C^r(M)$– Linear, More precisely, if $f, g \in C^r(M)$ and X and Y are vector fields on M , then $\sigma\ (f\ X + g\ Y) = f\sigma\ (X) + g\sigma\ (Y)$. For these functions are equal at each $p \in M$.

Corollary 2.3.4

Using the notation above let $\sigma = \sum_{i=1}^{n} \alpha_i\ \tilde{w}^i$ on V , and let $F^*(\sigma) = \sum_{j=1}^{m} \beta_j w^j$ on U ,where ∞_v and β_j are functions on V and U respectively, and $\tilde{w}^i, w^j$ are the coordinate co frames. Then $F^*(\tilde{w}^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} w^j$ and $\beta_j = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j}\ \alpha_i \circ F$. For $i = 1,......,n$ and $j = 1,.....,m.$ The first formulas give the relation of the bases; the second those of the components. If we apply this directly to a map of an open subset of R^m into R^n , these give for $F^*(dy^i)$ the formula

(8) $$\left\{ \ F^*(dy^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j}\ dx^j \ \right\} \quad , \quad i = 1,......,n$$

2.4 The Spectrum of the Laplacian in Riemannian Manifolds

To any compact Riemannian manifold (M,g) is boundary we associate second- order (P.D.E) , the Laplace operator Δ is defined by $\Delta(f) = -div(grad\ f)$ for $f \in L^2(M, g)$. We also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with the set of eigenvalues of Δ is called the spectrum of Δ or of M which we will write as space Δ or space (M,g) they form a discrete sequence $0=\lambda_0 \le \lambda_1 \le .... \le \lambda_n$ for simplicity , we will assume that M is connected . This will for example imply that the smallest eigenvalue λ_0 occurs with multiplicity .

Definition2.4.1

If L is aliner operator defined on T_pM , then the spectrum of L is the set of eigenvalues of L .It is denoted by space (L) . We take the Laplace operator Δ defined as $\Delta = -(d\,\delta + \delta d)$, where δ is adjoin of d in spectral geometry we consider the following two equations :

(i) Does the spectrum of M determine the geometry of M .

(ii) Does the geometry of M determine the spectrum of M.

proposition 2.4.2

The geometry of Riemannian manifold completely determines the spectrum the metric determines the Laplace operator is spectrum .

Definition 2.4.3

Sequences occur can as the spectra of manifolds a version of this question. Has been answered what finite sequences can occur as the initial part of spectra of manifolds . If M is a closed connected manifold of dimension greater that or Equal to 3 , the any p reassigned finite sequence $0=\lambda_0 \le \lambda_1 \le .... \le \lambda_k$ is Sequence of first $k+1$ eigenvalues of Δ_g for some choice of the metric g on M . In particular , this means that for closed connected manifolds of dimension 3 or Greater , there are no restrictions on the multiplicities of the eigenvalues λ_i for $i \ge 0$. In 2-dimension , there are some restrictions on the multiplicities of the eigenvalues. Let M be a closed connected 2-manifold with Euler characteristic $\chi(M)$, and let m_j be the multiplicity of the $(j-th)$ eigenvalue $(j \ge 0)$ of the laplacian operator associated to a metric on M then . If M is the unit sphere, then $m_j \le 2_j + 1$. For finite sequences $0=\lambda_0 \le \lambda_1 \le .... \le \lambda_k$ however the result by-colin de verdiere holds – even in dimension 2 .

2.5 : Estimates on the first Eigenvalue

The geometry of a manifold affects more that the multiplicities of the eigenvaluees . Here we will focus on bounds on the first non-zero eigenvalue λ_1 imposed by the geometry the first lower bound is due to lich neowicz .

Theorem 2.5.1

Let (M,g) be a closed Riemannian manifold of dimension $n \ge 2$ and let Ric be its Ricci tensor field if .

(9) $$\text{Ric}(X,X) \ge (n-1), \quad k \ge 0$$

For some constant $k \ge 0$, and for all $X \in T(M)$, then $\lambda_1 \ge nk$.

Theorem 2.5.2

Let (M,g) be a closed Riemannian manifold , if $(X,X) \ge (n-1)k \ge 0$ Ric . For some nonnegative constant k and for all $X \in T(M)$ then.

(10) $$\left\{ \lambda_1 \ge \frac{(n-1)}{4} + \frac{\pi^2}{D^2(M)} \right\}$$

It is in general much easier to given upper bounds on λ_1 that it is give lower bounds . The basic result in this area is a comparison theorem due to a complete Riemannian n- manifold whose Ricci curvature is $\geq (n-1)k$, $k \geq 0$ is some const.

Definition 2.5.3
We mentioned a above that a metric g , defines an inner product not just on T_a but also an inner product g^* on T_a^* , with this we can define an inner product on pth
exterior power

(11) $$T_a^* \ \Lambda^P : (\alpha_1 \Lambda \alpha_2 \Lambda .... \Lambda \alpha_P \ , \beta_1 \Lambda \beta_2 \Lambda .... \Lambda \beta_P) = Det \ g^*(\alpha_i, \beta_i)$$

Thus if $dx_1 \Lambda dx_2 \Lambda ....\Lambda dx_n$ defines the orientation $w = \sqrt{\det g_{ij}} dx_1 \Lambda .... \Lambda dx_n$. On a compact manifold we can integrate this to obtain total volume – so a metric defines not only length but also volumes, Now take $\alpha \in \Lambda^P T_a^*, \beta \in \Lambda^{n-P} T_a^*$ and define $f_\beta : \Lambda^P T_a^* \rightarrow R$, *by* $f_\beta(\alpha) w = \beta \Lambda \alpha$ .But we have an inner product , so any liner map on $\Lambda^P T_a^*$ is of the form $\alpha \rightarrow (\alpha, \gamma)$ for some $\gamma \in \Lambda^P T_a^*$ so we have a well –defined liner map $\beta \rightarrow \gamma\beta$ form $\Lambda^{n-P} T_a^* to \Lambda^P T_a^*$ satisfying $(\gamma_\beta, \alpha) w = \beta \Lambda \alpha$.

Definition 2.5.4 :[Hodge Star Operator]
The Hodge star operator is the linear map $* : \Omega^P(M) \rightarrow \Omega^{n-P}(M)$ with the property that at each point.

(12) $$(\alpha, \beta) \ w = \alpha \Lambda * \beta$$

Proposition 2.5.5
Let M be an oriented Riemannian manifold with volume for w , and let $\alpha \in \Omega^P(M), \beta \in \Omega^{P-1}(M)$ be forms of compact support then .

(13) $$\int_M (d^* \alpha, \beta) w = \int_M (\alpha, d\beta) \ w$$

Definition 2.5.6
Let M be an oriented Riemannian manifold , then the Laplacian on p-forms is the deferential operator $\Delta : \Omega^P(M) \rightarrow \Omega^P(M)$ defined by .

(14) $$\Delta : (dd^*) + (d^* d)$$

Definition 2.5.7
A differential form $\alpha \in \Omega^P(M)$ is harmonic if $\Delta\alpha = 0$ .On a compact manifold harmonic ply a important role, which there is no time to explore in this course
Here is the starting point.

Definition 2.5.8
Let M be a compact oriented Riemannian manifold then .
(i). a p-form is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$
(ii) In each de Rham co-homology class there is at most one harmonic from.

Theorem 2.5.9
If M is a compact n-manifold with Ricci curvature $\geq (n-1)(-k)$, $k \geq 0$ then

(15) $$\left\{ \lambda_1 \leq \frac{(n-1)^2}{4} k + \frac{c^2}{D^2(M)} \right\}$$

Where c_2 is positive constant depending only on N.

2.6 Geometric Implications Of The spectrum

The spectrum does not in general determine the geometry of a manifold Neverthless earthiness , some geometric information can be extracted from the spectrum . In what follows , we define a spectral invariant to be any thing that is completely determined by the spectrum .

Definition 2.6.1

Let M be a Riemannian manifold . A heat kernel or alternatively fundamental solution to the heat equation , is a function $k:(0,\infty)\times(M\times)M\rightarrow M$. That satisfies $k(t,x,y)$ is c^1 in t and c^2 in x and y , $\frac{\partial k}{\partial t}+\Delta_2(k)=0$ where Δ_2 is the Laplacian with respect to the second variable .

(16) $$\lim_{t\rightarrow 0^{\pm}} \int_M k(t,x,y)\ f(y)\,dy = f(x)$$

for any compactly supported function f on M .The heat kernel exists and unique for Riemannian manifold , its importance stems from the fact that the solution to the heat equation

(17) $$\left\{ \frac{\partial u}{\partial t}+\Delta(u)=0\ ,\ u:[0,\infty]\times M\rightarrow R \right\}$$

.

Where Δ is Laplacian with respect to second variable , with initial condition $u\,(0,x)=f(x)$ is given by:

(18) $$u(t,x)=\int_M k(t,x,y)\,f(y)\,dy$$

If $\{\lambda_i\}$ in spectrum of M and $\{\zeta_i\}$ are the associated eigenfunctions (normalized so the they form an orthonormal basis of $L^2(M)$ then we can write $k(t,x,y)=\sum_i e^{-\lambda_i t}\zeta_i(x)\zeta_i(y)$ $spec(m_1/G,g)=spec\ (m_2/G,g)$ from this it clear that the heat trace $Z(t)=\int_M K(t,x,x)=\sum_i e^{-\lambda_i t_i}$ spectral invariant . The heat trace has an asymptotic expansion as $t\rightarrow 0^{\pm}$.

(19) $$Z(t)=(4\pi t)^{\dim M/2}\sum_i^{\infty} a_j t^j$$

Where the a_j are integrals over M of universal homogenous polynomials in the curvature and covariant derivatives. The first few of these are : $a_0=vol(M)$

(20) $$\left\{ a_1=\frac{1}{6}\int_M S\ ,a_2=\int_M (5S^2-2|Ric|^2-|Rm|^2 \right\}$$

Where S is the scalar curvature , Ric : is the Ricci tensor , Rm : is the curvature tensor . the dimension the volume and total scalar curvature are thus completely determined by spectrum . If M is a surface then the Gauss Bonnet theorem implies that the Euler characteristic of M is also a spectral invariant . Amore in depth study of the heat trace can yield more information of dimension $n\leq 6$ and if M has same spectrum as the n-sphere S^n with the standard metric (resp . RP^m) then M M is in fact isometric to S^n (resp . RP^n) more on this can be found .

Definition 2.6.2

As was alluded to earlier, geometry is not in general a spectral in variant two manifolds are said to be isospectral if they have the same spectrum . Of non isometric isospectral manifolds was found too distinct but isospectral manifolds .

2.7 [Direct Computation of The Spectrum]

The first of those is straightforward: direct computation . it rarely possible to explicitly compute the spectrum of a manifold were actually discovered via this method . Milnor's example mentioned above consists of two isospectral factory-quotients of Euclidean space by lattices of full rank being one of full rank being one of the few examples of Riemannian manifolds whose spectra can be computed explicicitly spherical space forms – quotients of spheres by finite groups of orthogonal transformations acting without fixed points form another class of examples of manifolds is spectral for the Laplaction acting on p-forms for $R \le k$ but not for the Laplaction acting on p-forms for $R \le k+1$ (recall that a lens space is spherical space form where the group is cyclic .

2.8 Intrinsic Ultracontractivily on Bounded Bomains Manifolds

We first consider on R^d let D be aconnected bounded lipschitz domain in $R^{d\,n}(d \ge 1)$. And Δ with laplacian with Dirichlet boundarg conditions on D . It is well Know that the spectrum of Δ is discrete , $\sigma(-\Delta) = (\lambda_i) \ge 1$ with $0 \le \lambda_1 \le \lambda_2 \le .....$, and each λ_i is an eigenvalue with finite multiplicity . Denote by P_t^D The dirichlet heat kernel on D , and $\phi \ge 0$ the first normalized eigenfunction of $-\Delta$ and D it is also well known that D is intrinsically ultracontractive (i.e).

(21) $$\left\{ \zeta_t = e^{\lambda_1 t} \sup_{x,y\in D} \frac{p_i(x,y)}{\phi(x)\Psi(y)} \langle \infty, t \rangle \right\}$$

Indeed ,this is given true for more general domains such as holder domains of order o . The main purpose of this section is to clarify the short time behavior of ζ_t For lipschitz domains . when D is a … domain .

(22) $$\zeta_t \le 1 + C_t^{-(d+2)/2} \ , \ t \ge 0$$

Holds for some constant ≥ 0 this estimate was extended recently to smooth compact Riemannian manifolds (under some additional) geometrical assumptions) our aim is to study similar estimate for less smooth domains D . we shall see that the estimate ,holds for $C^{1,\alpha}$ domains for any $\alpha \ge 0$, If D is metrely lipchitzian (i.e) $C^{1,0}$ is no larger true . for instance , for $D = (0,1)^d$ one has $\phi(x) = \prod_{k=1}^{d} P_t^{(0,1)}(x_i, y_i)$, $\phi(x) = \prod_{k=1}^{d} \sin(\pi, x_i)$. and where $\sin(\pi r)$ is the first dirichlet eigenfunction on [0,1] . thus combining this with below for, $D = (0,1)$ we obtain .

(23) $$\frac{1}{2} t^{-3d/2} \le \zeta_t \le C\, t^{-3d/2} \ , \ t \in (0,1]$$

For some constant $C \ge 0$. A natatural question is there fore whether for lipschitz domain there exists a constant $C \ge 0$ such that :

(24) $$\zeta_t \le 1 + Ct \ , \ t \ge 0$$

We shall see that the answer is no, in general .It is true that $\zeta_t \le 1 + Ct^{-p}$. for some (qualitative) constant of the boundary .

We prove that for any $B \ge 0$,there exists alipschitz (connected) domain D such that $t^B \zeta_t$ is not bounded $t \to 0$ we summarize this as well as the large time behavior and a lower that domain D ia called Lipschitzian if for any $x \in \partial D$. there exist $S \ge 0$ a coordinate system is called (Lipschilzian) , $(r,\theta) \in R \times R^{d-1}$, and a Lipschitz function f on R^{d-1} such that x is the origin and .

(25) $$\left[\begin{array}{l} B(x,s) \cap D = B(x,s) \cap \{(r,\theta) : r \ge f(\theta)\} \\ B(x,s) \cap \partial D = B(x,s) \cap \{(r,s) : r = f(\theta)\} \end{array} \right]$$

A Lipschitz domain is called $C^{1,\alpha}$ for some $\alpha \ge 0$, if the corresponding Lipschitz function satisfies .

(26) $$\left|\nabla f(a) - V(b)\right| \le C\left|a-b\right|^{\alpha}$$

for some $C \ge 0$ and for all, $a,b \in R^{d-1}$. In this definition it is required that ≥ 2 , if $d=1$, D is an open bounded interval .

Theorem 2.8.1

If D is a $C^{1,\alpha}$ - domain for some ≥ 0 , there exists a constant $C \ge 0$, such that .

(27) $$\left\{ \quad \max\left\{1, \frac{1}{C}t\right\}^{-(a+2)/2} \le \zeta_t \le 1 + C(\wedge t)e^{-(d+2)} , \qquad for\, all \qquad t \ge 0 \right\}$$

For any $B \ge 0$, there exists a bounded Lipschitz domain $D \subset R^2$ such that $\lim_{t\to 0} Supt^B \zeta_t = +\infty$

Now let M be a d-dimensional connected Riemannian manifolds and D an open bounded $C^{1,1}$ domain in M . then for any $x \in \partial D$ there exist $S \ge 0$, a local coordinate system in $(r,\theta) \in R \times R^{d-1}$ in $B(x,s)$. (The open geodesic ball at x with radius ,s) and $f \in C_b^1(R^{d-1})$ with bounded second derivatives such that holds. For any.

(28) $$y = (r,\theta) \in B(x,s) \cap \overline{D} \quad \text{define} \quad f(y) = r - f(\theta) \ge 0$$

Then $y = (r,\theta) \in B(x,s) \cap \overline{D}$ has bounded seconded order derivatives furthermore there exists aconstant $C \ge 0$ such that .

(29) $$p(y) \le C\left|(r,\theta) - f(\theta,\theta)\right| = cF(y)$$

Where ρ is the Riemann $\rho \ge \rho_1$, $on\, D$ distance to ∂D .This by the partition of unity, there exists a non negative function $\overline{\rho} \in C_b^1(\overline{D})$ with bounded derivative and $\rho\,\big|_{\partial D} = 0$ such that :

(30) $$\rho \ge \rho_1 \ , on\, D$$

For some constant $1 \ge 0$, since $\overline{D}$ is compact for simplicity we may and do assume that M is compact to .

(31) $$L = N\sum_{i=1} X_i^2 + X$$

Where X is a bounded measurable vector field and $\{X_i\}_{i=1}^N$ are C^1 vector fields we assume that L is elliptic that is.

(32) $$\left\{ \begin{array}{l} (f,f) = \sum_i^N (X_i, f)^2 \ge |\nabla f|^2 \ , \ f \in C^1 \\ \mu(f^2) \le r\,\mu(|\nabla f|^2) + B(r)\,\mu(\psi)|f|^2, r \ge 0, f, f \in C^1(M) \end{array} \right\}$$

For some constant $2 \ge 0$. Thus under a local coordinate system on has .

(33) $$L = \sum_{i=1}^d a_{ij}\partial_i\partial_j + \sum_{i=1}^d b_i\partial_i$$

Where $(a_{ij})_{d\times d}$ is continous and strictly positive definite $b_i(1 \le i \le d)$ are bounded measurable . The L-diffusion process uniquely exists . For any $\in D$, Let $(X_t(x))$ be the L-diffusion process starting from x and , $T(x) = \inf\{t \ge 0, X_t(x) \in \partial D\}$ .For all bounded measurable function f on D .To study the (ontrnsic ultracontractivity) of PD_t . We assume that L is symmetric w.r.t a probability measure $\mu(dx)$, $\mu(dx) = 1, D^{v(x)}dx$ where V is abounded .Measurable function and (dx) the Riemannian volume measure by the elasticity and the sobolev in equality , we know that spectrum of L on D with dirichlet boundary condition is discrete , As in section 1, we let $\lambda_1 \ge \lambda_2$ be the first two dirichlet eigenvalues and $\phi \ge 0$ the normalized first eigen-function .

Theorem 2.8.2

Let $D \subset M$ be an open bounded $C^{1,1}$ domain and L a symmetric second order elliptic operator for some bounded measurable vector field X and C^1 - vector fields $\{X\}_{i=1}^{N}$ such that holds . Then for ζ_t defined in with the present λ_1, ϕ and the transition density μ .

III. Operators completes Riemannian manifolds

Let M be complete ,connected ,non – compact Riemannian manifold of dimension d . Let $L = \Delta + \nabla V$ for some $\in C^2(M)$. Then L generates a unique (Dirichlet) diffusion semi group P_t on M which is symmetric in $L^2(M)$, where $\mu = e^{V(x)}dx$. for dx the Riemannian volume measure . A assume that $\lambda_1 = \inf \sigma(-L)$ is an eigenvalue of $-L$. Since M is connected λ_1 has a unique unite eigenfunction $\theta \geq 0$. In order to study the intrinsic ultracontractivity P_t we make use of the following intrinsic super – poincare inquality introuduced .

(34) $$\mu(f^2) \leq r\mu\left(|\nabla f|^2\right) + \beta(r)\mu(\Psi)|f|^2 \quad , C^1(M) \quad r \geq 0, f \in C^1(M)$$

Which is equivalent to $\varepsilon(f,g) = \mu\ (\ \langle \nabla f, \nabla g \rangle\ )$, let Ric denote the curvature and ricci curvature on M respectively let ρ be the Riemannian distance on M , and simply write $\rho_0 = \rho(0,\cdot)$ for fixed reference point $0 \in M$ .Let k and K be two positive increasing function on $[0,\infty)$ such that :

(35) $$\sec \leq -k_0\,\rho_0 \ ,\ Ric \geq -k_0\rho_0 \quad ,\ \rho_0 \geq 1$$

Holds on M . here $\leq -k_0\rho_0$. means that for any $x \in M$ and unit vectors $X,Y \in T_x$ with $\langle X,Y \rangle = 0$, one has sec $(X,Y) \leq k$ or $(X,Y) \leq -k(\rho_0(x))$ while $Ric \geq -\ k_0\rho_0$ means that $Ric(X,Y) \geq -k(\rho_0(x)|X|^2)$ for any $x \in M$ and $X \in T_x$ for a positive increasing function h , on $(0,\infty)$ we let

(36) $$h^{-1}(r) = \inf\{s \geq 0, h(s)\ \} \quad r \geq 0$$

Theorem 3.1.1 :

Let M be a carton –Hadamard manifold with ≥ 2 , and let $L = \Delta$ A assume that (3,15) holds for some positive incrasing functions k with $k(\infty) = \infty$. We have $\sigma_{ess}(L) = \phi$ holds with .

(37) $$\beta(r) = \theta r^{\frac{-d}{2}} \exp\left[\theta k^{-1}(\theta / r)\ \sqrt{k(4 + 2k^{-1}(\theta)}\ \right]$$

For some constant $\theta \geq 0$ If $k^{-1}(R)\sqrt{k(4+2k^{-1}(R)} \leq CR^{\varepsilon}$, $R \geq 1$. Holds for some constants $C \geq 0$ and $(0,1)$, then P_t in intrinsically ultra contractive with .If , holds for comfort some $C \geq 0$ and $\varepsilon = 1$, then P_t is intrinsically hyper contractive , if $\geq -k$ for some constant $k \geq 0$, then $\sigma_{ess}(\Delta) \neq 0$ since M is non-compact and complete , this follows from a comparison for the first Dirichlet eignvalue and the Donnelly $-L_i$ decomposition principle for essential spectrum . $\inf_{ess}(-\Delta) \leq \sup_{x \in M} \ \lambda_1\ (B(x,1) \leq \lambda_1(k)$. Where $\lambda_1\ B(x.1)$ is first Dirichlet eigenvalue of $-\Delta$ on D and Where $\lambda_1(k)$ is one on unite geodesic ball in the d-dim . parabolic space with Ricci curvature equal to k .Thus the assumption $k(\infty) = \infty$ is also reasonable . Next , we consider the case with drift. To this end , we adopt the following Bakry – Emery curvature $Ric_{m,l}$. Instead of Ric . A assume that for some ≥ 0 , and positive increasing function k on has instead of the second condition in

(38) $$\left\{\ Ric_{l,m} = Ric - H_{ess_v} - \frac{\nabla_v \otimes \nabla_v}{m} \geq -k_0\rho\ \right\}_0$$

Moreover , let r be a positive increasing function on $(0,\infty]$ such that $L_{\rho_0} \geq \sqrt{r_0}\rho_0$, $\rho_0 \geq 1$

Theorem 3.1.2

Let S pole in M such that hold for some increasing positive functions k and $r(\infty)=\infty$ then $\sigma_{ess}(L)=0$ moreover ,assuming .

(39) $$\left\{ \lim_{\rho_0(x)\to\infty} \frac{\sqrt{k(2+2_{\rho_0}(x))}}{\log^{+}\mu(B(x.1)} = 0 \right\}$$

Where $B(x,1)$ is the unit geodesic ball at , we have holds with .

(40) $$B(r)=\theta r^{-(m+d+1)/2}\exp\left\lfloor \theta r^{-1}(32/r)\sqrt{k(4)} \right\rfloor, \theta \geq 0$$

For there some constant $\theta \geq 0$, If there exist $C \geq 0$ and $\varepsilon \in (0.1)$ such that $r^{-1}(R)\sqrt{k(2+2r^{-1}(R))} \leq CR^{\varepsilon}$, $R \geq 1$ then P_t is intrinsically ultra contractive and) holds for some constant $C \geq 0$ .If holds for some $C \geq 0$ and $\varepsilon = 1$ then P_t is intrinsically hyper contractive .

Example 3.1.3

let M be a Cartan-Hadamard manifold with $Ric \geq -C(\rho_0^{2(-1)})$ For some constants $C \geq 0$ and α >1, let $v=\theta\rho_0$ for some constant $\theta \geq 0$ and $\rho_0 \geq 1$,then $\sigma_{ess}(L)=0$ and holds $C \geq 0$ For some constant $C \geq 0$ consequently . P_t is is intrinsically ultra contractive if and only if $\theta_1,\theta_2 \geq 2$, and when $\left\| P_t^{\theta} \right\|_{L^2(\mu\theta)} \to L^{\infty}_{M\theta} \leq \theta_1 \exp\left[\theta_1 t^{-/-2}\right]$,$t \geq 0$. For some constants $\theta_1,\theta_2 \geq 0$, which is sharp in the sense that constant θ_2 can not be replaced by any positive function $\theta_2(t)\downarrow 0$ as $t \downarrow 0$. P_t is intrinsically hyper contractive if and only if $\theta_1,\theta_2 \geq 2$.

3.2 The Spectral Geometry of operators of Dirac and Laplace Type

We have also given in each a few additional references to relevant . The constraints of space have of necessity forced us to omit many more important references that it was possible to include and we a apologize in a dance for that . We a the following notational conventions , let (M,g) be compact Riemannian manifold of dim. M with boundary ∂M Let Greek indices μ,γ range from 1 to m , and index a local system of coordinates $x=(x^1,...........,x^m)$ on the interior of M expand the metric in the form $dS^2 = g_{\mu\gamma}dx^{\mu}dx^{\gamma}$ were $g_{\mu\gamma} = (\partial_{x_\mu},\partial_{x_\gamma})$ and where we a adopt the Einstein convention of summing over repeated indices we let $g^{\mu\gamma}$ be the inverse matrix the Riemannian measure is given by $dx=(dx^1,...........,dx^m)$ for $g=\sqrt{\det(g_{\mu\gamma})}$ let ∇ be the levi-Civita connection. We expand $\nabla_{\partial x,\gamma}.\partial_{x\mu} = \Gamma_{\gamma\mu}\sigma\partial x_{\sigma}$ .Where $\Gamma_{\gamma\mu}\sigma$ are the m R are may then be given by.

(41) $$R(X,Y)=\left[\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right]$$

and given by $R(X,Y,Z,W)=g(R(X,Y),Z,W)$. We shall let Latin indices (i,j) range from 1 to m and index a local orthonormal frame $\{e_1,..........,e_m\}$ for the components of the curvature tensor scalar curvature τ Are then given by setting $\mathrm{P}_{ij} = R_{ijkk}$ *and* $\tau = \mathrm{P}_{ij} = R_{ikki}$. We shall often have an auxiliary vector bundle set V and an auxiliary given on V , we use this connection and the " Levi-Civita" connection to covariant differentiation , let dy be the measure of the induced metric on boundary ∂M , we choose a local orthonormal from near the boundary M , so that $\{e_m\}$ is the inward unit normal . We let indices (a,b) range from 1 to m-1 and index the induced local frame $\{e_1,......,e_{m-1}\}$ for the tangent bundle f the boundary , let

$L_{a,b} =_{g} \left(\nabla_{e_a} eb, e_m\right)$ denote the second fundamaental form . we some over indices with the implicit range indicated . Thus the geodesic curvature K_g is given by $K_g = L_{aa}$. We shall let denote multiple tangential covariant differentiation with respect to the " Levi-Civita" connection the boundary the difference between and being of coursemeasured by the fundamental form.

3.3 The Geometric of Operators of Laplace and Dirac Type
In this section we shall establish basic definitions discuss operator of Laplace and of Dirac type introduce the De-Rham complex and discuss the Bochner Laplacian and the weitzenboch formula . Let D be a second of smooth sections $C^{\infty}(v)$ of a vector bundle v over space M, expand.

(42) $$D = -\left\{ a^{\mu\nu} \partial x_{\mu} \partial x_{\nu} + a^{\sigma} \partial x_{\sigma} + b \right\}$$

where coefficient $\left\{ a^{\mu\nu}, a^{\mu}, b \right\}$ are smooth endomorphism's of v, we suppress the fiber indices . We say that D is an operator of Laplace type if A^2. On $C^{\infty}(v)$ is said to be an operator of Dirac type if A^2 is an operator of laplace operator of Dirac type if and only if the endomorphisms γ^{ν} satisfy the Clifford commutation relations .

(43) $$\gamma^{\nu}\ \gamma^{\mu} + \gamma^{\mu}\ \gamma^{\nu} = -2g^{\mu\nu}(id).$$

Let A be an operator of Dirac type and let $\zeta = \zeta_{\nu} dx^{\nu}$ be a smooth 1-form on M we let $\gamma(\zeta) = \zeta_{\nu} v^{\nu}$ define a Clifford module structure on V . This is independent of the particular coordinate system chosen . We can always choose a fiber metric on V so that γ is skew adjoint . We can then construct a unitary connection ∇ on V so that $\nabla\gamma = 0$ such that a connection is called compatible the endomorphism if ∇ is compatible we expand $A = \gamma\ \nabla^{\nu}_{\partial x_{\nu}} + \psi_A$, ψ_A is tensorial and does not depend on the particular coordinate system chosen it does of course depend on the particular compatible connection chosen.

Definition 3.3.1 The De-Rham Complex
The prototypical example is given by the exterior algebra , let $C^{\infty}\left(\Lambda^p M\right)$ be the space of smooth p forms. Let $d : C^{\infty}\left(\Lambda^p M\right) \to C^{\infty}\left(\Lambda^{p-1} M\right)$ be exterior differentiation if ζ is cotangent vector , Let $ext(\zeta) : w \to \zeta \Lambda w$ denote exterior multiplication and let $\operatorname{int}(\zeta)$ be the Dual , Interior multiplication, $v(\zeta) = ext(\zeta) - \operatorname{int}(\zeta)$ define module on exterior algebra $\Lambda(M)$. Since $d + \delta = v\left(dx^{\nu}\right)\nabla_{\partial x_{\nu}}$. $d + \delta$ is an operator of Diract type the a associated laplacian.

(44) $$\Delta_m = (d + \delta)^2 = \left(\Delta^0_m \oplus ......\oplus \Delta^p_m \oplus .....\oplus \Delta^m_M\right)$$

decomposes as the Direct sum of operators of laplace type Δ^p_m on the space of smooth p forms $C^{\infty}\left(\Lambda^p M\right)$ on has $\Delta^0_M = -g^{-1}\partial x_{\mu} g g^{\mu\nu} \partial x_{\nu}$ it is possible to write the p-form valued Laplacian in an invariant form . Extend the " Levi-Civita" comection to act on tensors of all types .Let $\widetilde{\Delta}_{M^w} = -g^{\mu\nu} w$, $\mu\nu$ define Bochner or reduced Laplacian , let R given the associated action of curvature tensor . The Weitzenbock formula terms of the Buchner Laplacian in the form

(45) $$\Delta_M = \widetilde{\Delta}_{M^w} + \frac{1}{2}\gamma\left(dx^{\mu}\right)\gamma\left(dx^{\gamma}\right)R_{\mu\gamma}$$

This formalism can be applied more generally.

Theorem 33.2

Let u be p-regular solution of satisfying as . Assume also that $u \le M$ on $\partial\Omega$ for some constant $M \ge 0$. Then $u^+ \in L^{\infty}(\Omega)$and there exists a universal constant $C = C(n, p, |\Omega|) \ge 0$ such that .
$u \le M + C\left(\| u^+\|_p + a + b + k\right)$ *in* Ω, where $k = k\ (a_1, b_1, b_2)$is given by.

(46) $$k = \left\{\left[b_1 + (a_2 + b_2)^{1/p}\right]^{n/p} + \left(a_2^{1/p} + b_2^{1/(p-1)}\right)M \qquad if\ p \ge 0\right.$$

The same result can be given if we depend on the x-variable with some regularity precisely denoting simply by $\|f\|$the norm in $L^q(\Omega)$of f and q is assigned ,we have the following generalization.

Theorem 3.3.3

Let U be p-regular solution in Ω where A and B satisfy (with $b_1 = b_2 = 0$, suppose $u \le M$ on $\partial\Omega$ for some constant $M \ge 0$ and if $p = 1$we also assume that $a_2 + b \le |\Omega|^{\frac{1}{n}}(1-\delta)/S$

Defection 3.3.4

For a sub harmonic function on f on Riemannian manifold M if there exist a pints in M at which attains the this property is to give a certain condition for a sub harmonic function to be constant , when we give attention to the fact relative t these maximum principles.

Definition 3.3.5 Liouvile's

(i)Let f be a sub harmonic function on R^n, if it is bounded then it is constant.

(ii) Let f be a harmonic functions on R^n , $m \ge 3$. If it is bounded then it is constant . We are interested in Riemannian analogues of Liouvile,s theorem compared with these Last tow theorems we give attention to the fact that there is an essential difference between base manifold . In fact one is compact and the other is complete and an compact , we consider have a family of Riemannian manifold (M, g) at the global situations it suffices to consider a bout the family of complete Riemannian manifold of course , the subclass of compact Riemannian manifolds. (M, g) is complete Riemannian manifold since a compact Riemannian manifold .

Theorem 3.3.6 Complete Riemannian Manifold

A let M be complete Riemannian manifold whose Ricci curvature is bounded from below , if C^2- nonnegative function f satisfies Where Δ denotes the Laplacian on M , then f vanishes identically, the purpose of this theorem is t prove the following (Leadville Type) theorem in a complete Riemannian manifolds similar to theorem in a complete Riemannian manifold similar to give anther proof of (Nishikawas theorem) . In this note main theorem is as follows .

Theorem 3.3.7
Let M be a complete Riemannian manifold whose Ricci curvature is bounded from blew , if C^2 - nonnegative function f satisfies $\Delta f \le C_0 f_n$. Where C_0 is any positive constant and n is any real number greater f vanishes identically .

Theorem 3.38 Ricci Riemannian Manifold
Let M an n-dimensional Riemannian manifold whose Ricci curvature is bonded from below on M , Let G be a C^2 - functions bounded from below on M , then for any $\varepsilon \ge 0$, there exists a point p such that

(47) $$|\nabla G(P)| \le \varepsilon \quad , \quad \Delta G(P) - \varepsilon \ and \ \inf G + \varepsilon \ge G(p)$$

Proof :
In this section we prove the theorem stated in introduction first all in order prove theorem , then our theorem is directly obtained as a corollary of this property and hence Nishikawas theorem is also a direct consequence of this (Nishikawas one)

Theorem 3.3.8 Manifold and Ricci Curvature
Let M be a complete Riemannian manifold whose Ricci Curvature is bounded from blew , Let F be any formula of the variable F with constant coefficients such that $F(f) = (C_0 f^n + C_1 f^{n-1} + ... + C_k f^{n-k}) + C_{k+1}$ Where $n \ge 1$, $1 \ge n-k \ge 0$ and $C_0 \ge C_{k+1}$ if a C^2 - nonnegative function f satisfies .

(48) $$\Delta f \ge F(f)$$

Then we have Where f_1 denotes the super mum f the given function f .

Proof :
From the assumption there exists a positive number a which satisfies $C_{k+1} \le a^n C_0$ For the constant a given above the function $G(f)$ with respect to 1-variable f is defined by $(f+a)^{\frac{1-n}{2}}$, n is the maximal degree of the f , then it is easily seen that G is the C^2 - function so that it is bounded from appositive by the constant $a^{\frac{1-n}{2}}$ and bounded from below by 0 , By the simple calculating we have

(49) $$\left\{ \nabla G = -\frac{n-1}{2} G^{\frac{n+1}{n-1}} \nabla f \right\}$$

Hence we get by using the above equation $\left(\frac{1-n}{2}\right) G \left(\frac{2n}{n-1}\right) \Delta f = G\Delta G - \left(\frac{n+1}{n-1}\right) |\nabla G|^2$

Since the Ricci curvature is bounded from below by the assumption and the function G defined above satisfies the condition that it is bounded from below , we can apply the theorem to the function G . Given any positive number ε there exist a point P at which it satisfies the following relationship at P .

(50) $$\left\{ \left(\frac{1-n}{2}\right) G(P)^{\frac{2n}{n-1}} \Delta(f) \ge -\varepsilon G(P) - \left(\frac{n+1}{n-1}\right) \varepsilon^2 \right\}$$

Can be derived , where $G(P)$ denotes $G(f\varphi)$ thus for any convergent sequence e $G_0 = \inf G$, by taking a sub sequence , if necessary because the sequence is bounded and therefore each term

$G(P_m)$ of the sequence satisfies equation we have $G(P_m) \to G_0 = \inf G$ and the assumption $n \ge 1$.An the other hand it follows from we have

(51) $$\left\{ \left(\frac{1-n}{2}\right) G(P_m)^{\frac{2n}{n-1}} \Delta(P_m) \ge -\varepsilon_m G(P_m) - \left(\frac{n+1}{n-1}\right) \varepsilon_m{}^2 \right\}$$

And the right side of the a above inequality converges to zero because the function G is bounded by choosing the constant a it satisfies $C_{k+1} a^{-n} \le C_0$, A accordingly there is a positive number δ such that $\left(\frac{1-n}{2}\right) C_{k+1} a^{-n} \le \delta \le \left(\frac{n+1}{2}\right) C_0$, C_0 is the constant coefficient of the maximal degree of function F so for a given such that $a\delta \ge 0$, we can take a sufficiently large integer m such that

(52) $$\left(\frac{1-n}{2}\right) G(P_m)^{\frac{2n}{n-1}} F(f(P_m)) \ge -\delta$$

Where we have used the assumption equation of and equation so this inequality together with the definition of $G(P_m)$ Yield $F(f(P_m)) \le \left(\frac{2\delta}{n-1}\right)(f+a)\,(P_m)^n$

Remark 3.3.9

Suppose that a nonnegative function f satisfies the condition we can directly yield $\nabla f^{\,n-1} = (n-1) f^{\,n-2} \nabla f$

(53) $$\Delta f^{\,n-1} = (n-1)(n-2) f^{\,n-3} \nabla(f \nabla f) + (n-1) f^{\,n-2} \Delta f$$

we define a function h by $f^{\,n-1}$, if $n \ge 2$ then it satisfies $\Delta h \ge (n-1) C_0 h^2$ Thus concerning the theorem in the case $n \ge 2$ the condition is equivalent $1 \le n \le 2$ where C_1 is a positive constant

Lemma 3.3.10

Let D be an operator of Laplace type on a Riemannian manifold, there exists a unique connection ∇ on V and there exists a unique endomorphism E of V , so that $D\phi = -\phi_{ii} - E\phi$ if we express D locally in the form $D = \{g^{\mu\gamma} \partial x_\gamma \partial x_\mu + a^\mu \partial x_\mu + b\}$ then the connection 1-form w of ∇ and the endomorphism E are given by .

(54) $$\left\{ w_\gamma = \frac{1}{2}\left(g_{\gamma\mu} a^\mu + g^{\sigma E} \Gamma_{\sigma E_\gamma} id\right) \; and \; E = b - g^{\gamma\mu}\left(\partial x_\gamma w_\mu + w_\gamma w_\mu - w_\sigma \Gamma^\sigma{}_{\gamma\mu}\right) \right\}$$

Let V be equipped with an auxiliary fiber metric, then D is self-adjoin if and only if ∇ is unitary and E is self-adjoin we note if D is the Spinor bundle and the Lichnerowicz formula with our sign convention that $E = -\frac{1}{4} J(id)$ where J is the scalar curvature.

Definition 3.4 Heat Trace Asymptotic for closed manifold

Throughout this section we shall assume that D is an operator of Laplace type on a closed Riemannian manifold (M,g) . We shall discuss the L^2 - spectral resolution if D is self adjoin , define the heat equation introduce the heat trace and the heat trace asymptotic present the leading terms in the heat trace .Asymptotic references for the material of this section and other references will be cited as needed , we suppose that D is self-adjoin there is then a complete spectral resolution of D on $L^2(v)$. This means that we can find a complete orthonormal basis $\{\phi_n\}$ for $L^2(v)$ where the ϕ_n are a smooth sections to V which satisfy the equation $D\phi_n = \lambda_n \phi_n$.

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Definition 3.4.1

Let V be a vector space and $\varphi \in V$ are tensors. The product of φ and ψ, denoted $\varphi \otimes \psi$ is a tensor of order $(r+s)$ defined by $\varphi \otimes \psi(v_1,..v_r,..v_{r+1},...v_{r+s}) = \varphi(v_1,....,v_r)\psi(v_{r+1},....,v_{r+s})$.

The right hand side is the product of the values of φ and ψ.The product defines a mapping $(\varphi,\psi) \to \varphi \otimes \psi$ of $x^r(V) \to {}^{r+s}(V)$.

CONCLUSION :

The basic notions on Laplace geometry calculus , on geometric formulation of the notion of the differential and the inverse function φ^{-1} A certain familiarity with the elements of the differential Geometry of surfaces with the basic definition of differentiable manifolds , starting with properties of covering

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