Coloring of Intuitionistic Fuzzy Directed Hypergraphs

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Abstract

A hypergraph is a set \( V \) of vertices and a set \( E \) of non-empty subsets of \( V \), called hyperedges. Unlike graphs, hypergraphs can perform higher-order interactions in social and communication networks. Directed hypergraphs are much like directed graphs. Colors are used to distinguish the classes. Coloring a hypergraph \( H \) must assign at least two different colors to the vertices of every hyperedge. That is, no edge is monochromatic. In this paper, \( p \)-coloring, \( K \)-coloring, \( p \)-chromatic number, spike and spike reduction of intuitionistic fuzzy directed hypergraph (IFDHG), skeleton of spike reduction are studied. Further, a few properties of coloring of intuitionistic fuzzy directed hypergraph are discussed. Also, it has been proved that in an ordered IFDHG, a primitive coloring is a \( K \)-coloring of the IFDHG. **Keywords:*** Intuitionistic fuzzy directed hypergraph, coloring of IFDHG, properties.

1 Introduction

Fuzzy sets (FSs) introduced by L.A.Zadeh in 1965 [12] are generalization of crisp sets. K.T.Atanassov introduced the concept of intuitionistic fuzzy sets (IFSs) in 1999 [1] as an extension of FSs. These sets include not only the membership of the set but also the non-membership of the set along with degree of uncertainty. In order to expand the application base, the notion of a graph was generalized to that of a hypergraph. In 1976, Berge [2] introduced the concepts of graph and hypergraph. In [3], the concepts of fuzzy graph and fuzzy hypergraph were introduced.

In this way, the authors got motivated to expand the concepts such as \( p \)-coloring, \( K \)-coloring, \( p \)-chromatic number, spike and spike reduction of intuitionistic fuzzy directed hypergraph, skeleton of spike reduction were studied.

The paper has been organized as follows: Section 2 deals with the definitions of fuzzy hypergraph, intuitionistic fuzzy hypergraph and intuitionistic fuzzy directed hypergraph and the notations used. In section 3, the concepts of \( p \)-coloring, \( K \)-coloring, \( p \)-chromatic number, spike and spike reduction of IFDHG, skeleton of spike reduction are studied. Section 4 concludes the paper.

2 Preliminaries

In this section, definitions of intuitionistic fuzzy set, intuitionistic fuzzy hypergraph and IFDHG are dealt with.

**Definition 2.1.** [1] Let a set \( E \) be fixed. An **intuitionistic fuzzy set (IFS)** \( V \) in \( E \) is an object of the form \( V = \{ (v_i, \mu_i(v_i), \nu_i(v_i)) / v_i \in E \} \), where the function \( \mu_i : E \rightarrow [0, 1] \) and \( \nu_i : E \rightarrow [0, 1] \) determine
the degree of membership and the degree of non-membership of the element \( v_i \in E \), respectively and for every \( v_i \in E \), \( 0 \leq \mu(v_i) + \nu(v_i) \leq 1 \).

**Definition 2.2.** [4] Let \( E \) be the fixed set and \( V = \{(v_i, \mu_i(v_i), \nu_i(v_i)) | v_i \in V \} \) be an IFS. Six types of Cartesian products of \( n \) subsets \( V_1, V_2, \cdots, V_n \) of \( V \) over \( E \) are defined as

\[
V_1 \times V_2 \times V_3 \times \cdots \times V_n = \{(v_1, v_2, \cdots, v_n), \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \nu_i | v_i \in V_1, v_2 \in V_2, \cdots, v_n \in V_n \},
\]

\[
V_1 \times V_2 \times 2 V_3 \times 2 V_4 \times \cdots \times 2 V_n = \{(v_1, v_2, \cdots, v_n), \sum_{i=1}^{n} \mu_i - \sum_{i \neq j} \mu_i \mu_j + \sum_{i \neq j} \mu_i \mu_j \mu_k \cdots \mu_n + \sum_{i \neq j \neq k \cdots \neq n} \mu_i \mu_j \mu_k \cdots \mu_n \}
\]

\[
V_1 \times \nu_2 \times V_3 \times 3 V_4 \times 3 V_5 \times \cdots \times 3 V_n = \{(v_1, v_2, \cdots, v_n), \prod_{i=1}^{n} \nu_i \nu_j + \sum_{i \neq j} \nu_i v_j \nu_k \cdots \nu_n + \sum_{i \neq j \neq k \cdots \neq n} \nu_i \nu_j \nu_k \cdots \nu_n \}
\]

\[
V_1 \times 4 V_2 \times 4 V_3 \times 4 V_4 \times 4 V_5 \times \cdots \times 4 V_n = \{(v_1, v_2, \cdots, v_n), \min(\mu_1, \mu_2, \cdots, \mu_n), \max(\nu_1, \nu_2, \cdots, \nu_n) | v_i \in V, v_2 \in V_2, \cdots, v_n \in V_n \}
\]

\[
V_1 \times 5 V_2 \times 5 V_3 \times 5 V_4 \times 5 V_5 \times \cdots \times 5 V_n = \{(v_1, v_2, \cdots, v_n), \sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \nu_i | v_i \in V, v_2 \in V_2, \cdots, v_n \in V_n \},
\]

It is used to determine the edge membership \( \mu_{ij} \) and the edge non-membership \( \nu_{ij} \).

**Definition 2.3.** [5] An intuitionistic fuzzy hypergraph (IFHG) is an ordered pair \( H = (V, E) \) where

(i) \( V = \{v_1, v_2, \cdots, v_n\} \), is a finite set of intuitionistic fuzzy vertices,

(ii) \( E = \{E_1, E_2, \cdots, E_m\} \) is a family of crisp subsets of \( V \), respectively and

(iii) \( E_j = \{(v_i, \mu_j(v_i), \nu_j(v_i)) | v_i \in V, \mu_j(v_i), \nu_j(v_i) \geq 0 \) and \( \mu_j(v_i) + \nu_j(v_i) \leq 1 \}, j = 1, 2, \ldots, m \),

(iv) \( E_j \neq \emptyset, j = 1, 2, \ldots, m \),

(v) \( \bigcup_j \supp(E_j) = V, j = 1, 2, \ldots, m \).

Here, the hyperedges \( E_j \) are crisp sets of intuitionistic fuzzy vertices, \( \mu_j(v_i) \) and \( \nu_j(v_i) \) denote the degrees of membership and non-membership of vertex \( v_i \) to edge the \( E_j \). Thus, the elements of the incidence matrix of IFHG are of the form \( (v_{ij}, \mu_j(v_i), \nu_j(v_i)) \). The sets \( (V, E) \) are crisp sets.

**Notations:**

1. Hereafter, \( \langle \mu(v_i), \nu(v_i) \rangle \) or simply \( (\mu, \nu) \) denote the degrees of membership and non-membership of the vertex \( v_i \in V \), such that \( 0 \leq \mu_i + \nu_i \leq 1 \).

2. \( \langle \mu(v_{ij}), \nu(v_{ij}) \rangle \) or simply \( (\mu_{ij}, \nu_{ij}) \) denote the degrees of membership and non-membership of the edge \( (v_i, v_j) \in V \times V \), such that \( 0 \leq \mu_{ij} + \nu_{ij} \leq 1 \).

Also \( \mu_{ij} \) is the membership value of \( i \)-th vertex in \( j \)-th edge and \( \nu_{ij} \) is the non-membership value of \( i \)-th vertex in \( j \)-th edge.

**Note:**

The support of an IFS \( V \) in \( E \) is defined as \( \supp(E_j) = \{v_i/ \mu_j(v_i) > 0 \) and \( \nu_j(v_i) > 0 \} \).

**Definition 2.4.** [6] An intuitionistic fuzzy directed hypergraph (IFDHG) \( H \) is a pair \( (V, E) \), where \( V \) is a non empty set of vertices and \( E \) is a set of intuitionistic fuzzy hyperarcs; an intuitionistic fuzzy hyperarc \( E_i \in E \) is defined as a pair \( (t(E_i), h(E_i)) \), where \( t(E_i) \in V \), with \( t(E_i) \neq \emptyset \), is its tail, and \( h(E_i) \in V - t(E_i) \) is its head. A vertex \( s \) is said to be a source vertex in \( H \) if \( h(E) \neq s \), for every \( E_i \in E \). A vertex \( d \) is said to be a destination vertex in \( H \) if \( d \neq t(E) \), for every \( E_i \in E \).

**Definition 2.5.** [11] Let \( H = (V, E) \) be an IFDHG and \( H^{r_i,s_i} = (V^{r_i,s_i}, E^{r_i,s_i}) \) be the \( (r_i, s_i) \) - level intuitionistic fuzzy hypergraph of \( H \). The sequence of real numbers \( \{r_1, r_2, \ldots, r_n; s_1, s_2, \ldots, s_n \} \), such that \( 0 \leq r_i \leq h_i(H) \) and \( 0 \leq s_i \leq h_i(H) \), satisfying the
properties:
(i) If \( r_i < \alpha \leq 1 \) and \( 0 \leq \beta < s_i \) then \( E^{\alpha,\beta} = \emptyset \),
(ii) If \( r_{i+1} \leq \alpha \leq r_i \) ; \( s_i \leq \beta < s_{i+1} \) then \( E^{\alpha,\beta} = E^{r_i,s_i} \),
(iii) \( E^{r_i,s_i} \subset E^{r_{i+1},s_{i+1}} \)
is called the fundamental sequence of \( H \), and is denoted by \( F(H) \).
The core set of \( H \) is denoted by \( C(H) \) and is defined by \( C(H) = \{ H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n} \} \). The corresponding set of \( \{ r_i, s_i \} \) - level hypergraphs \( H^{r_1,s_1} \subset H^{r_2,s_2} \subset ... \subset H^{r_n,s_n} \) is called the \( H \) induced fundamental sequence and is denoted by \( I(H) \). The \( \{ r_n, s_n \} \)- level is called the support level of \( H \) and the \( H^{r_n,s_n} \) is called the support of \( H \).

**Definition 2.6.** [11] Let \( H = (V,E) \) be an intuitionistic fuzzy directed hypergraph and \( C(H) = \{ H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n} \} \). \( H \) is said to be ordered if \( C(H) \) is ordered. That is \( H^{r_1,s_1} \subset H^{r_2,s_2} \subset ... \subset H^{r_n,s_n} \). The intuitional fuzzy directed hypergraph is said to be simply ordered if the sequence \( \{ H^{r_i,s_i} \} / i = 1, 2, 3, ..., n \) is simply ordered. That is, if \( H \) is ordered and if whenever \( E \in H^{r_1,s_1} - H^{r_1,s_1} \) then \( E \not\subseteq H^{r_1,s_1} \).

3 Coloring of intuitionistic fuzzy directed hypergraphs
Throughout this section, \( H \) refers to an IFDHG \( H = (V,E) \).

**Definition 3.1.** Let \( H \) be an IFDHG. A primitive \( p \)-coloring \( A \) of \( H \) is a partition \( \{ A_1, A_2, A_3, ..., A_p \} \) of \( V \) into \( p \)-subsets (colors) such that the support of each intuitional fuzzy hyperedge of \( H \) intersects atleast two colors of \( A \), except spike edges.

**Definition 3.2.** Let \( H \) be an IFDHG. Let \( C(H) = \{ H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n} \} \). An \( K \)-coloring \( A \) of \( H \) is a partition \( \{ A_1, A_2, A_3, ..., A_p \} \) of \( V \) into \( p \)-subsets (colors) such that \( A \) induces a coloring for each core hypergraph \( H^{r_i,s_i} \) of \( H \) with \( H^{r_i,s_i} = (V_i,E_i) \) where \( V_i \subset V \) and \( E_i \subset E \). The restriction of \( A \) to \( V_i \), \( \{ A_1 \cap V_i, A_2 \cap V_i, A_3 \cap V_i, ..., A_k \cap V_i \} \), is coloring of \( \{ H^{r_i,s_i} \} \). (Allow color set \( A_1 \) to be empty).

**Example 1.** Consider an IFDHG, \( H \) with \( V = \{ v_1, v_2, v_3, v_4, v_5 \} \) and \( E = \{ E_1, E_2, E_3, E_4 \} \) whose adjacency matrix as follows:

\[
H = \begin{pmatrix}
E_1 & E_2 & E_3 & E_4 \\
v_1 & (0.8, 0) & (0.8, 0) & (0.1) & (0.1) \\
v_2 & (0.8, 0) & (0.8, 0) & (0.8, 0) & (0.1) \\
v_3 & (0.7, 0.1) & (0.1) & (0.1) & (0.7, 0.1) \\
v_4 & (0.1) & (0.1) & (0.6, 0.3) & (0.6, 0.3) \\
v_5 & (0.3, 0.2) & (0.3, 0.2) & (0.1) & (0.1)
\end{pmatrix}
\]

The IF core hypergraphs of \( H \) are as follows:

\( H^{0.8,0} = \{ \{ v_1, v_2 \}, \{ v_2 \} \} \)

\( H^{0.7,0.1} = \{ \{ v_1, v_2, v_3 \}, \{ v_1, v_2 \}, \{ v_2 \}, \{ v_3 \} \} \)

\( H^{0.6,0.3} = \{ \{ v_1, v_2, v_3 \}, \{ v_1, v_2 \}, \{ v_2, v_4 \}, \{ v_3, v_4 \} \} \)

\( H^{0.3,0.2} = \{ \{ v_1, v_2, v_3, v_4 \}, \{ v_1, v_2, v_5 \}, \{ v_2, v_4 \}, \{ v_3, v_4 \} \} \)

The corresponding graph is shown in Figure 1.

Suppose \( A = \{ \{ v_1, v_2 \}, \{ v_4 \}, \{ v_3, v_5 \} \} \)

Then \( A \) is a coloring of \( H^{0.6,0.3} \) and \( H^{0.3,0.2} \) but not \( H^{0.8,0} \). Hence \( A \) is a \( K \)-coloring of \( H \) with intensity \( (0.8, 0) \).

**Definition 3.3.** The \( p \)-chromatic number of an IFDHG \( H \) is the minimal number \( \chi_p(H) \), of colors needed to produce a primitive coloring of \( H \). The chromatic number of \( H \) is the minimal number, \( \chi(H) \), of colors needed to produce a \( K \)-coloring of \( H \).

**Example 2.** Consider an IFDHG, \( H \) where \( V = \{ v_1, v_2, v_3, v_4, v_5, v_6 \} \) and \( E = \{ E_1, E_2, E_3, E_4, E_5, E_6, E_7 \} \) with adjacency matrix as below:

The corresponding graph is shown in Figure 2.
Hence $A$ and $H$ of $\emptyset$ where $E$ and $E^3$, $E^2$, $E^4$, $E^5$, $E^6$, $E^7$ are defined by

$$
\begin{pmatrix}
    0.6, 0.3 & 0.6, 0.3 & 0.6, 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\
    0.6, 0.3 & 0.1 & 0.1 & 0.6, 0.3 & 0.6, 0.3 & 0.6, 0.3 & 0.1 \\
    0.1 & 0.6, 0.3 & 0.6, 0.3 & 0.6, 0.3 & 0.1 & 0.1 & 0.1 \\
    0.5, 0.2 & 0.1 & 0.1 & 0.5, 0.2 & 0.5, 0.2 & 0.5, 0.2 & 0.5, 0.2 \\
    0.1 & 0.1 & 0.2, 0.1 & 0.1 & 0.1 & 0.2, 0.1 & 0.2, 0.1 \\
    0.1 & 0.5, 0.2 & 0.5, 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\
\end{pmatrix}
$$

Then $C(H) = \{H^{r_1}, s_1 = (V^{r_1}, s_1, E^{r_1})| i = 1, 2, 3, 4\}$. where

$$(r_1, s_1) = (0.6, 0.3); (r_2, s_2) = (0.5, 0.2); (r_3, s_3) = (0.3, 0.1); (r_4, s_4) = (0.2, 0.1)$$

$E_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$

$E_2 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}\}$

$E_3 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}\}$

$E_4 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}\}$

$E_5 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}\}$

$E_6 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}\}$

$E_7 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}\}$

Consider $H^{r_1, s_1}$. Suppose $A_1, A_2$ is a coloring of $H^{r_1, s_1}$. Then $\{v_1, v_2\} \cap A_1 \neq \emptyset, \{v_1, v_3\} \cap A_1 \neq \emptyset, \{v_2, v_3\} \cap A_1 \neq \emptyset \text{ for } i = 1, 2$.

Hence $A_1 \cap A_2 \neq \emptyset$, a contradiction. Thus $\chi(H^{r_1, s_1}) = 3$.

$$\chi(H^{r_1, s_1}) = 3$$

$\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6, v_7\}\}$ is a coloring of $H^{r_2, s_2}$, so $\chi(H^{r_2, s_2}) = 2$

For $H^{r_3, s_3}$, since $E \subseteq V, |E| = 3$. Hence $\chi(H^{r_3, s_3}) = 3$ and $\chi(H^{r_4, s_4}) = 3$.

**Definition 3.4.** A spike reduction of $E_i \in F_{\mu}(V)$, denoted by $E_i$, is defined as $E_i(v_i) = \max_i{\{r_i, s_i\}}/|E^{r_i, s_i}| \geq 2, (0 \leq r_i \leq E^{r_i, s_i}, 0 \leq s_i \leq E^{r_i, s_i})$.

**Note:** i) If $A = \emptyset$, then $E(v_i) = 0$.

(ii) If $E_i$ is a spike, then $E_i = \chi_0$

**Theorem 3.5.** Let $H$ be an IFDHG and let $\hat{H} = (\tilde{V}, \tilde{E})$, where $\tilde{E} = \{E_i| E_i \in E\}$ and $\tilde{V} = \bigcup_{\hat{E} \in \tilde{E}} \text{supp}(\hat{E})$.

**Example 3.** Consider example 2, $(E_T)^{0.5, 0.2} = \{v_4\}$.

Hence $E_T(v_1) = E_T(v_2) = E_T(v_3) = 0$ and $E_T(v_4) = \hat{E}_T(v_5) = \hat{E}_T(v_6) = \hat{E}_T(v_7) = (0, 2, 0.1)$. It is clear that $E_T \neq E_T$. Since $E_T \neq \emptyset$, $E_T$ is not a spike.

**Note:** If each intuitionistic fuzzy hyperedge is a spike, then $\tilde{E} = \emptyset$. Hence $\hat{H}$ is not an IFDHG. Thus this concept cannot be proceeded in real coloring problem. So excluding it from further consideration and always proceed by assuming $\tilde{H}$ exists.

**Definition 3.6.** Let $H$ be an IFDHG and $A$ is...
a primitive coloring of $H$, then $A$ is a $\mathcal{K}$-coloring of $H$.

**Proof.** Since $H$ is an ordered IFDHG, from Definition 2.6, $C(H)$ is also an ordered IFDHG. That is, if $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$, then $H^{r_1,s_1} \subset H^{r_2,s_2} \subset ... \subset H^{r_n,s_n}$.

Since $A$ is a primitive coloring of $H$, there exists a partition of $V$ into $p$-subsets $\{A_1, A_2, A_3, ..., A_p\}$ such that $A$ induces a coloring for each core hypergraph, $H^{r_i,s_i}$ of $H$. Hence $A$ is a $\mathcal{K}$-coloring of $H$.

**Theorem 3.2.** Let $H$ be an IFDHG and suppose $C(H) = \{H^{r_i,s_i} | i = 1, 2, 3...n\}$, where $0 \leq r_i \leq h_p(H)$ and $0 \leq s_i \leq h_i(H)$. If $H^{r_i,s_i}$ is a simple IFDHG and singleton hyperedges do not appear in any core hypergraph of $H$ and if each primitive coloring $A$ of $H$ is a $\mathcal{K}$-coloring of $H$, then $H$ is an ordered IFDHG.

**Proof.** It is known that $H^{r_i,s_i} = (V^{r_i,s_i}, E^{r_i,s_i})$ for $1 \leq i \leq n$. Assume $H^{r_n,s_n}$ is simple and that $H$ is not ordered. Then there exists a primitive coloring of $H$ that is not a $\mathcal{K}$-coloring of $H$.

**Construction:**
Since $H$ is not ordered, there exists some core hypergraph $H^{r_i,s_i}$, where $i \leq n - 1$, such that some hyperedges $E_i' \in E_i$ is not an edge of $E_j, j > i$.

From definition 2.5, there is an intuitionistic fuzzy hyperedge $E_i \in E$ such that $E_i^{r_i,s_i} = E'_i$.

Let $E_i' = E_i^{r_i+1,s_i+1}$ and $F = E_i^{r_n,s_n}$. Then $E_i' \subset E_i \subset F$.

Since $H^{r_n,s_n}$ is simple and $F \in E_n$, it follows that $E_i' \notin E_n$. Hence $|E_i'| \geq 2$.

Hence, there is a primitive coloring of $H$ that is not a $\mathcal{K}$-coloring of $H$.

**Theorem 3.3.** Let $H$ be an ordered IFDHG and $C(H) = \{H^{r_i,s_i} | i = 1, 2, 3...n\}$, then $\chi(H^{r_i,s_i}) \leq \chi(H^{r_2,s_2}) \leq \cdots \leq \chi(H^{r_n,s_n}) = \chi(H)$, where $\chi(H^{r_i,s_i})$ represents the minimal number of colors required to color the crisp hypergraph $H^{r_i,s_i}$.

**Definition 3.6.** Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be a pair of IFDHGs such that $V_1 \subseteq V_2$.

Suppose $A' = \{A_1, A_2, A_3, ..., A_p\}$, where $\bigcup_{i=1}^p A_i = V_1$ and $A_i \neq \emptyset$, for $i = 1, 2, ..., p$ is a $\mathcal{K}$-coloring or $(p$-coloring) of $H_1$. Then $A''$ is a stable $\mathcal{K}$-coloring or $(p$-coloring) extension of $A'$ to $H_2$ if $A'' = \{A_1', A_2', A_3', ..., A_p'\}$ is a $\mathcal{K}$-coloring or $(p$-coloring) of $H_2$ which satisfies

i) $\bigcup_{i=1}^p A_i' = V_2$

ii) $A_i \subseteq A_i'$ for $i = 1, 2, ..., p$. 

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Figure 2: Intuitionistic fuzzy directed hypergraph with $(a)\chi(H^{r_2,s_2}) = 2$ and $(b)\chi(H^{r_3,s_3}) = 3$.
4 Skeleton of transversals of intuitionistic fuzzy directed hypergraph \((H^s)\).

Let \(H\) be an IFDHG with fundamental sequence \(F(H) = \{r_1, r_2, \ldots, r_n; s_1, s_2, \ldots, s_n\}\) where \(0 \leq r_i \leq h_y(H)\) and \(0 \leq s_i \leq h_y(H)\) and core set \(C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, \ldots, H^{r_n,s_n}\}\).

**Construction 1:** The construction of \(\widehat{C(H)}\) from \(C(H)\) is a recursive process:

**Step 1:** Determine a IF partial hypergraph \(\widehat{H}^{r_1,s_1}\) of \(H^{r_1,s_1}\) by eliminating all the IF hyperedge in \(H^{r_1,s_1}\) that properly contain another edge of \(H^{r_1,s_1}\).

**Step 2:** Eliminate all IF hyperedges of \(H^{r_2,s_2}\) which are either properly contained another edge of \(H^{r_2,s_2}\) or contains (properly or improperly) an IF hyperedges of \(H^{r_2,s_2}\). Then either all edges of \(H^{r_2,s_2}\) are eliminated or the remaining edges form an IFDHG \(\widehat{H}^{r_2,s_2}\) of \(H^{r_2,s_2}\).

**Step i:** For \(i = 1, 2, 3, \ldots k\) where \(1 \leq k \leq n-1\) and \(n \geq 2\) this process is repeated.

**Step \(k+1\):** Eliminate all IF hyperedges of \(H^{r_{k+1},s_{k+1}}\) or contain an IF hyperedge of \(\widehat{H}^{r_1,s_1}\) for \(i = 1, 2, 3, \ldots k\) if \(k\) exists. Then, either all edges of \(H^{r_{k+1},s_{k+1}}\) are eliminated (and \(\widehat{H}^{r_{k+1},s_{k+1}}\) does not exists) or the remaining IF hyperedges form a partial hypergraph \(\widehat{H}^{r_{k+1},s_{k+1}}\) of \(H^{r_{k+1},s_{k+1}}\). Continuing recursively up to \(n\), we obtain \(F(\widehat{H}) = \{r_1^*, r_2^*, \ldots, r_n^*, s_1^*, s_2^*, \ldots, s_n^*\}\) of \(F(H)\). The IF coreset \(C(H) = \{\widehat{H}^{r_1^*,s_1^*}, \widehat{H}^{r_2^*,s_2^*}, \ldots, \widehat{H}^{r_n^*,s_n^*}\}\) of IF partial hypergraph form \(C(H)\).

**Note:** Each member of \(\widehat{C(H)}\) has non-empty edge set and that for every \(\langle r_i, s_i \rangle \in F(\widehat{H}) \setminus \{r_1^*, r_2^*, \ldots, r_n^*, s_1^*, s_2^*, \ldots, s_n^*\}\) the entire core hypergraph \(H^{r_i,s_i}\) was eliminated in the recursive process.

**Definition 4.1.** The skeleton of \(\widehat{H}\), denoted by \(\widehat{H}^\square\), is defined as \(H^\square = (\widehat{H})^s\).

**Theorem 4.1.** Let \(H\) be an IFDHG and suppose for each \(H\) there exists a \(\widehat{H}\), then every \(p\)-coloring of \(H^\square\) is a \(K\)-coloring of \(H^\square\) and conversely.

**Proof.** Since \(H^\square\) is an ordered IFDHG, the result follows directly from Theorem 3.1.

**Theorem 4.2.** Let \(H\) be an IFDHG and there exists \(\widehat{H}\), then every \(K\)-coloring of \(H\) is a color stable extension of some \(p\)-coloring of \(H^\square\). Conversely, any extended \(K\)-coloring of \(H^\square\) without adding any color is a color stable extended \(K\)-coloring of \(H\).

**Example 4.** Consider an IFDHG, \(H\) with \(V = \{v_1, v_2, v_3, v_4\}\) and \(E = \{E_1, E_2, E_3, E_4, E_5\}\) whose adjacency matrix is as given below:

The IF core hypergraphs are as follows:

\[
\begin{align*}
\langle r_1, s_1 \rangle &= (0, 0); \langle r_2, s_2 \rangle = (0.7, 0.2); \langle r_3, s_3 \rangle = (0.4, 0.2); \langle r_4, s_4 \rangle = (0.3, 0.2) \\
H^{0.9,0} &= \{\{v_1\}\} \\
H^{0.7,0.2} &= \{\{v_1, v_2\}, \{v_2\}\} \\
H^{0.4,0.2} &= \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\} \\
H^{0.3,0.2} &= \{\{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_3\}, \{v_2, v_3, v_4\}\}
\end{align*}
\]

No edge of \(H^{r_1,s_1}\) properly contains another edge of \(H^{r_2,s_2}\). Hence \(\widehat{H}^{r_1,s_1} = H^{r_1,s_1}\). For \(H^{r_2,s_2}\), \(\{v_1, v_2\} \supseteq \{v_1\}\). Thus, removing \(\{v_1, v_2\}\) from \(E^{r_2,s_2}\) gives \(v_2\). Hence \(\widehat{H}^{r_2,s_2} = \{v_2\}\). For \(H^{r_3,s_3}\), \(\{v_2, v_3\} \supseteq \{v_2\}\). Removing edges which are properly contained in \(\widehat{H}^{r_2,s_2}\) gives \(\widehat{H}^{r_3,s_3} = \{v_1, v_3\}\).

It follows that \(\langle r_1^*, s_1^* \rangle = (r_1, s_1)\); \(\langle r_2^*, s_2^* \rangle = (r_2, s_2)\) and \(\langle r_3^*, s_3^* \rangle = (r_3, s_3)\). Then \(H^s = (V^s, E^s)\) where \(V^s = \{v_1, v_2, v_3, v_4\}\) and \(E^s = \{\{v_1\}, \{v_1, v_2\}, \{v_1, v_3\}\}\)

**Example 5.** In Example 1, \(\widehat{H} = H\) and so \(H^\square = H^s\). Every \(K\)-coloring of \(H\) is a color stable extension of some \(K\)-coloring of \(H^\square\). But every \(K\)-coloring of \(H^\square\) is a \(K\)-coloring of \(H\). Since \(E_1 = \{\{v_1\}\} = E_1^\square\) \(E_2 = \{\{v_2\}\}\) and \(E_2 = E_2^\square \cup \{v_1, v_2\}\).

**Example 6.** Let \(H\) be an IFDHG. In Example 4, \(H^s = (V^s, E^s)\) where \(V^s = \{v_1, v_2, v_3, v_4, v_5\}\) and \(E^s = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}\). Hence \(\{v_1\}, \{v_2\}, \{v_1, v_3\}\) are \(K\)-coloring of \(H^s\). Clearly chromatic number \(\chi(H^s) = 2\). Note that \(\{v_1, (0, 0)\}\) and \(\{v_2, (0.7, 0.2)\}\) are spikes in \(H^s\).
Consider spike reduction, $\tilde{H} = (\tilde{V}, \tilde{E})$ where
$\tilde{V} = \{v_1, v_2, v_3, v_4\}$ and $\tilde{E} = \{E_1, E_2, E_3, E_4, E_5\}$
which is represented by the adjacency matrix in Example 4:

Thus
$$H^{0.9,0} = \{\{v_1\}\}$$
$$H^{0.7,0.2} = \{\{v_1, v_2\}, \{v_2\}\}$$
$$H^{0.4,0.2} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$$
$$H^{0.3,0.2} = \{\{v_1, v_2\}, \{v_1, v_2, v_4\}, \{v_2, v_3\}, \{v_1, v_3, v_4\}\}$$

Then $H^{0} = (V^{0}, E^{0})$ where $V^{0} = \{v_1, v_2, v_3, v_4\}$ and $E^{0} = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$.

Hence $\{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$ are $K$-coloring of $H^{0}$.

Clearly $\chi(H^{0}) = 2$.

5 Conclusion

In this paper, an attempt has been made to study the coloring on IFDHG. Also, some interesting properties of IFDHGs are dealt with $p$-coloring, $K$-coloring, $p$-chromatic number, spike, spike reduction and skeleton of spike reduction. Further, it has been proved that if $H$ is an ordered IFDHG and $A$ is a primitive coloring of $H$, then $A$ is a $K$-coloring of $H$ and some other properties have also been analysed.

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References


