Interval Valued Intuitionistic Hesitant Fuzzy Weighted Einstein Arithmetic Averaging Operators

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Abstract:

Hesitant Fuzzy Set is a very useful tool in the situations where there are some difficulties in determining the membership of an element to a set. In this paper, we study the interval valued intuitionistic hesitant fuzzy information aggregation operators based on Einstein operational laws. The Einstein based interval valued intuitionistic hesitant Einstein fuzzy weighted averaging ($IVIHFWA_{\omega}^{\varepsilon}$) operator and interval valued intuitionistic hesitant Einstein fuzzy ordered weighted averaging ($IVIHFOWA_{\omega}^{\varepsilon}$) operator are proposed for aggregating interval-valued intuitionistic hesitant fuzzy information. The desirable properties of the $IVIHFWA_{\omega}^{\varepsilon}$ and $IVIHFOWA_{\omega}^{\varepsilon}$ operators such as idempotency, boundeness and monotonicity are discussed.

Keywords: Intuitionistic Hesitant Fuzzy Sets; Interval-valued Intuitionistic Hesitant Fuzzy Set; Interval valued intuitionistic hesitant Einstein fuzzy weighted averaging operator; interval valued intuitionistic hesitant Einstein fuzzy ordered weighted averaging operator.

1. Introduction:

In many decision making problems, crisp data are sometimes unavailable due to the fuzziness or vagueness of data in the domain of the problem. Several extensions of fuzzy sets (FS) by Zadeh (1965), namely, type 2 fuzzy set (Mizumoto and Tanaka, 1976) and intuitionistic fuzzy set (IFS) (Atanassov, 1986) have been introduced in the literature. The IFS is equivalent to interval-valued fuzzy sets (Atanassov and Gargov, 1989), and the prominent characteristic of IFS is that it assigns to each element a degree of membership and a degree of non-membership. These extensions cannot deal with the situation when decision makers have certain hesitancy in providing their preferences over the available alternatives. To deal with such cases, Torra (2010) introduced another generalization of fuzzy set, viz. hesitant fuzzy set (HFS) allowing the membership degree to have a set of possible values. Torra (2010) also discussed the relationships between hesitant fuzzy set and other three kinds of fuzzy sets (intuitionistic fuzzy sets, type-2 fuzzy sets and fuzzy multisets). More and more multiple attribute decision making theories and methods under hesitant fuzzy environment have been developed. Xia and Xu (2011) proposed some aggregation operators for hesitant fuzzy information, investigated the connections of these operators and applied them to the multi-criteria decision making. Xu and Xia (2011) gave a detailed study on distance and similarity measures for hesitant fuzzy sets and hesitant fuzzy elements respectively. Wei (2012) developed the hesitant fuzzy prioritized operators and applied them to multiple attribute decision making.

Lately, research on aggregation methods and multiple attribute decision making theories under hesitant fuzzy environment is very active and a lot of results have been obtained for hesitant fuzzy information. Xia et al. [2011] developed some confidence induced aggregation operators for hesitant fuzzy information. Xia et al. [2013] gave several series of hesitant fuzzy aggregation operators with the help of quasi arithmetic means. Wei [2012] explored several hesitant fuzzy prioritized aggregation operators and applied them to hesitant fuzzy decision making problems. Uma Maheswari.A and Kumari.P [2014] introduced the hesitant fuzzy Heronian Mean and interval-valued intuitionistic hesitant fuzzy Einstein geometric aggregation operators.

This paper is organized as follows: Section 2 introduces the basic concepts and operations on interval-valued intuitionistic hesitant fuzzy sets. In Section 3, we introduce a new arithmetic aggregation operator, i.e., interval valued intuitionistic hesitant fuzzy Einstein weighted arithmetic averaging ($IVIHFWA_{\omega}^{\varepsilon}$) operators based on Einstein operations. Some properties of the developed $IVIHFWA_{\omega}^{\varepsilon}$ operator are studied in detail. Section 4 introduces interval valued intuitionistic hesitant fuzzy Einstein ordered weighted arithmetic averaging operators ($IVIHFOWA_{\omega}^{\varepsilon}$). The final section offers some concluding remarks.

2. Preliminaries:

Definition 2.1 [Xia and Xu, 2011]

Let X be a fixed set, an Hesitant Fuzzy Set (HFS) on X is in terms of a function that when applied to X returns a subset of [0, 1]. Mathematically Xia and Xu express the HFS as,
$H = \{\langle x, h_H(x)\rangle / x \in X\}$ where $h_H(x)$ is a set of some values in [0, 1] denoting the possible membership degrees of the element $x \in X$ to the set H. $h_H(x)$ is called the hesitant fuzzy element (HFE)

Definition 2.2 IFS [K. T. Atanassov ,1986]

Let X be a fixed set. An IFS A in X is defined as $A = \{(x, \mu_A(x), \gamma_A(x) / x \in X)\}$ where μ_A and γ_A are mappings from X to the closed interval [0, 1] such that $0 \le \mu_A(x) \le 1$, $0 \le \gamma_A(x) \le 1$ and for all $x \in X$, $0 \le \mu_A(x) + \gamma_A(x) \le 1$ and they denote respectively the degree of membership and degree of non-membership of element $x \in X$ to the set A.

Sometimes, instead of exact values a range of values may be a more appropriate measurement to represent the vagueness. Atanassov and Gargov [1989] introduced the Interval Valued Intuitionistic Fuzzy Sets (IVIFS)

The interval valued intuitionistic hesitant fuzzy sets (IVIHFS) allows the membership of an element to be a set of several possible interval-valued intuitionistic fuzzy numbers.

Definition 2.3 [Z. Zhang,2013]

Let X be a fixed set, $E=\{\langle x,h_E(x)\rangle / x\in X\}$ where, $h_E(x)$ is a set of some IVIFNs in Ω denoting the possible membership and non-membership degree intervals of the element $x\in X$ to the set E.

$h=h_E(x)$ is called an interval valued intuitionistic hesitant fuzzy element (IVIHFE) and H denotes the set of all IVIHFEs. If $\alpha\in h$ then α is an IVIFN denoted by $\alpha=(\mu_\alpha,\gamma_\alpha)=\left([\mu_\alpha^L,\mu_\alpha^U],[\gamma_\alpha^L,\gamma_\alpha^U]\right)$.

The Einstein operation on IVIFNs is extended to IVIHFEs

Definition 2.4 [Uma Maheswari.A and Kumari.P]

Let $h=\left\{\left([\mu_\alpha^L,\mu_\alpha^U],[\gamma_\alpha^L,\gamma_\alpha^U]\right)/\alpha\in h\right\}$, $h_1=\left\{\left([\mu_{\alpha_1}^L,\mu_{\alpha_1}^U],[\gamma_{\alpha_1}^L,\gamma_{\alpha_1}^U]\right)/\alpha_1\in h_1\right\}$ and $h_2=\left\{\left([\mu_{\alpha_2}^L,\mu_{\alpha_2}^U],[\gamma_{\alpha_2}^L,\gamma_{\alpha_2}^U]\right)/\alpha_2\in h_2\right\}$ be three given IVIHFEs. Then the Einstein operation on them are defined as below.

(1) $h^c=\left\{\alpha^c/\alpha\in h\right\}=\left\{\left([\gamma_\alpha^L,\gamma_\alpha^U],[\mu_\alpha^L,\mu_\alpha^U]\right)/\alpha\in h\right\}$

(2)

$$h_1\oplus h_2=\left\{\left(\left[\frac{\mu_{\alpha_1}^L+\mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L\mu_{\alpha_2}^L},\frac{\mu_{\alpha_1}^U+\mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U\mu_{\alpha_2}^U}\right],\left[\frac{\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}{1+(1-\gamma_{\alpha_1}^L)(1-\gamma_{\alpha_2}^L)}\right],\left[\frac{\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}{1+(1-\gamma_{\alpha_1}^U)(1-\gamma_{\alpha_2}^U)}\right]\right)\Big/\alpha_1\in h_1,\alpha_2\in h_2\right\}$$

(3)

$$h_1\otimes h_2=\left\{\left(\left[\frac{\mu_{\alpha_1}^L\mu_{\alpha_2}^L}{1+(1-\mu_{\alpha_1}^L)(1-\mu_{\alpha_2}^L)},\frac{\mu_{\alpha_1}^U\mu_{\alpha_2}^U}{1+(1-\mu_{\alpha_1}^U)(1-\mu_{\alpha_2}^U)}\right],\left[\frac{\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}{1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L},\frac{\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}\right]\right)\Big/\alpha_1\in h_1,\alpha_2\in h_2\right\}$$

(4)For $\lambda > 0$

$$\lambda h_1 = \left\{ \left(\left[\frac{(1+\mu_{\alpha_1}^L)^\lambda - (1-\mu_{\alpha_1}^L)^\lambda}{(1+\mu_{\alpha_1}^L)^\lambda + (1-\mu_{\alpha_2}^L)^\lambda}, \frac{(1+\mu_{\alpha_1}^U)^\lambda - (1-\mu_{\alpha_1}^U)^\lambda}{(1+\mu_{\alpha_1}^U)^\lambda + (1-\mu_{\alpha_2}^U)^\lambda} \right], \left[\frac{2(\gamma_{\alpha_1}^L)^\lambda}{(2-\gamma_{\alpha_1}^L)^\lambda + (\gamma_{\alpha_1}^L)^\lambda}, \frac{2(\gamma_{\alpha_1}^U)^\lambda}{(2-\gamma_{\alpha_1}^U)^\lambda + (\gamma_{\alpha_1}^U)^\lambda} \right] \right) \middle/ \begin{matrix} \alpha_1 \in h_1 \\ \lambda > 0 \end{matrix} \right\}$$

(5)For $\lambda > 0$

$$(h)^\lambda = \left\{ \left(\left[\frac{2(\mu_{\alpha_1}^L)^\lambda}{(2-\mu_{\alpha_1}^L)^\lambda + (\mu_{\alpha_1}^L)^\lambda}, \frac{2(\mu_{\alpha_1}^U)^\lambda}{(2-\mu_{\alpha_1}^U)^\lambda + (\mu_{\alpha_1}^U)^\lambda} \right], \left[\frac{(1+\gamma_{\alpha_1}^L)^\lambda - (1-\gamma_{\alpha_1}^L)^\lambda}{(1+\gamma_{\alpha_1}^L)^\lambda + (1-\gamma_{\alpha_1}^L)^\lambda}, \frac{(1+\gamma_{\alpha_1}^U)^\lambda - (1-\gamma_{\alpha_1}^U)^\lambda}{(1+\gamma_{\alpha_1}^U)^\lambda + (1-\gamma_{\alpha_1}^U)^\lambda} \right] \right) \middle/ \alpha \in h \right\}$$

Definition 2.5 [Z. Zhang 2013]

Let $h_i \,(i = 1,2,...,n)$ be a collection of IVIHFEs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ the weight vector of $h_i \,(i = 1,2,...,n)$ with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. An interval-valued intuitionistic hesitant fuzzy weighted averaging (IVIHFWA) operator is a mapping $H^n \to H$ such that $IVIHFWA_\omega(h_1, h_2, ..., h_n) = \bigoplus_{i=1}^{n} \omega_i h_i$

$$= \left\{ \left(\left[1 - \prod_{i=1}^{n} (1-\mu_{\alpha_i}^L), 1 - \prod_{i=1}^{n} (1-\mu_{\alpha_i}^U)^{\omega_i} \right], \left[\prod_{i=2}^{n} (\gamma_{\alpha_i}^L)^{\omega_i}, \prod_{i=2}^{n} (\gamma_{\alpha_i}^U)^{\omega_i} \right] \right) \middle/ \alpha_i \in h_i \,(i = 1,2,...,\mathrm{n}) \right\}$$

Lemma 2.1 [V. Torra and Y. Narukawa(2007) ,,Z.S.Xu,2000]

Let $a_i > o, \omega_i > 0, i = 1,2,...n$ and $\sum_{i=1}^{n} \omega_i = 1$ then $\prod_{i=1}^{n} a_i^{\omega_i} \le \sum_{i=1}^{n} \omega_i a_i$ with equality if and only if $a_1 = a_2 = ... = a_n$

Definition 2.6 [Z. S. Xu, 2007]

If $\alpha = ([\mu_\alpha^L, \mu_\alpha^U], [\gamma_\alpha^L, \gamma_\alpha^U])$ is an IVIFN then the score function of α is defined as

$s(\alpha) = \frac{1}{2}(\mu_\alpha^L - \gamma_\alpha^L + \mu_\alpha^U - \gamma_\alpha^U)$ and the accuracy function of α is $h(\alpha) = \frac{1}{2}(\mu_\alpha^L + \gamma_\alpha^L + \mu_\alpha^U + \gamma_\alpha^U)$

Definition 2.7 [Z.Zhang, 2013]

For an IVIHFE $\tilde{h} = \{([\mu_\alpha^L, \mu_\alpha^U], [\gamma_\alpha^L, \gamma_\alpha^U])\}$ $s(\tilde{h}) = \sum_{\alpha \in \tilde{h}} s(\alpha) / \#\tilde{h}$ is called the score function of h, where

$h(\tilde{h}) = \sum_{\alpha \in \tilde{h}} h(\alpha) / \#\tilde{h}$ is called the accuracy function of h.

For an two IVIHFEs h_1 and h_2

(1) If $s(h_1) > s(h_2)$, then $h_1 > h_2$;

(2) If $s(h_1) = s(h_2)$, the following hold

(a) $h(h_1) > h(h_2)$, then $h_1 > h_2$

(b) $h(h_1) = h(h_2)$, then $h_1 = h_2$

(c) If $h(h_2) = h(h_1)$, then $h_2 > h_1$

3. Interval valued Intuitionistic Hesitant fuzzy Einstein weighted arithmetic averaging operators (IVIHFWA^ϵ)

In this section we define the new arithmetic aggregation operators based on Einstein operations.

Definition 3.1: Let $h_i = \left\{ \left(\left[\mu_{\alpha_i}^L, \mu_{\alpha_i}^U \right], \left[\gamma_{\alpha_i}^L, \gamma_{\alpha_i}^U \right] \right) / \alpha_i \in h_i \right\}$ $(i = 1, 2, ..., n)$ be a collection of IVIHFSs in L and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of h_i (i=1,2,…,n) such that $\omega_i \in [0,1]$ with $\sum_{i=1}^{n} \omega_i = 1$. Then an IVIHFWA^ϵ operator of dimension n is a mapping from L^n□□L such that

$$IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_n\right) = \omega_1 \tilde{h}_1 \oplus \omega_2 \tilde{h}_2 \oplus ... \oplus \omega_n \tilde{h}_n$$

If $\omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$ then IVIHFWG^ϵ reduces to interval valued intuitionistic hesitant fuzzy Einstein averaging(IVIHFA^ϵ) operator of dimension n which is defined as follows:

$$IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_n\right) = \frac{1}{n}\left(\tilde{h}_1 \oplus \tilde{h}_2 \oplus ... \oplus \tilde{h}_n\right)$$

Based on the Einstein operations we prove the following theorem.

Theorem 3.2:

Let $h_i = \left\{ \left(\left[\mu_{\alpha_i}^L, \mu_{\alpha_i}^U \right], \left[\gamma_{\alpha_i}^L, \gamma_{\alpha_i}^U \right] \right) / \alpha_i \in h_i \right\}$ $(i = 1, 2, ..., n)$ be a collection of IVIHFEs in L. Then their aggregated value by using IVIHFWA^ϵ operator is also an IVIHFE and

$$IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_n\right) = \left\{ \left(\left[\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^L\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^L\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^L\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^U\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^U\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^U\right)^{\omega_i}} \right], \right. \right.$$

$$\left. \left. \left[\frac{2\prod_{i=1}^{n}\left(\gamma_{\alpha_i}^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_i}^L\right)^{\omega_i} + \prod_{i=1}^{n}\left(\gamma_{\alpha_i}^L\right)^{\omega_i}}, \frac{2\prod_{i=1}^{n}\left(\gamma_{\alpha_i}^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_i}^U\right)^{\omega_i} + \prod_{i=1}^{n}\left(\gamma_{\alpha_i}^U\right)^{\omega_i}} \right] \right) \middle/ \alpha_i \in \widetilde{h_i}, (i = 1, 2, ..., n) \right\} \quad (1)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of h_i (i=1,2,…,n) such that $\omega_i \in [0,1]$ with $\sum_{i=1}^{n} \omega_i = 1$.

Proof: The first result follows from the operational law of Einstein. Let us prove equation (6) by using mathematical induction on n.

Equation (1) is obvious for n=1.Assume that the equation (1) is true for n = k. That is

$$IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_k\right)=\left\{\left(\left[\frac{\prod_{i=1}^{k}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i}-\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{k}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i}},\frac{\prod_{i=1}^{k}\left(1+\mu_{\alpha_i}^{U}\right)^{\omega_i}-\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^{U}\right)^{\omega_i}}{\prod_{i=1}^{k}\left(1+\mu_{\alpha_i}^{U}\right)^{\omega_i}+\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^{U}\right)^{\omega_i}}\right],\right.\right.$$

$$\left.\left.\left[\frac{2\prod_{i=1}^{k}\left(\gamma_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{k}\left(2-\gamma_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{k}\left(\gamma_{\alpha_i}^{L}\right)^{\omega_i}},\frac{2\prod_{i=1}^{k}\left(\gamma_{\alpha_i}^{U}\right)^{\omega_i}}{\prod_{i=1}^{k}\left(2-\gamma_{\alpha_i}^{U}\right)^{\omega_i}+\prod_{i=1}^{k}\left(\gamma_{\alpha_i}^{U}\right)^{\omega_i}}\right]\right)\middle/\alpha_i\in\widetilde{h_i},(i=1,2,\ldots,k)\right\} \quad (2)$$

Suppose n = k+1

$$IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_{k+1}\right)=IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_k\right)\oplus\omega_{k+1}\tilde{h}_{k+1}$$

$$=IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_k\right)\oplus\left\{\left(\left[\frac{\left(1+\mu_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}-\left(1-\mu_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}}{\left(1+\mu_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}+\left(1-\mu_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}},\frac{\left(1+\mu_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}-\left(1-\mu_{\alpha k+1}^{U}\right)^{\omega_{k+1}}}{\left(1+\mu_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}+\left(1-\mu_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}}\right],\right.\right.$$

$$\left.\left.\left[\frac{2\left(\gamma_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}}{\left(2-\gamma_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}+\left(\gamma_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}}},\frac{2\left(\gamma_{\alpha_i}^{U}\right)^{\omega_{k+1}}}{\left(2-\gamma_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}+\left(\gamma_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}}\right]\right)\middle/\alpha_i\in\widetilde{h_i},(i=1,2,\ldots,n)\right\}$$

Let $a_1^L=\prod_{i=1}^{k}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i},a_1^U=\prod_{i=1}^{k}\left(1+\mu_{\alpha_i}^{U}\right)^{\omega_i},b_1^L=\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i},b_1^U=\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^{U}\right)^{\omega_i}$

$c_1^L=\prod_{i=1}^{k}\left(\gamma_{\alpha_i}^{L}\right)^{\omega_i},c_1^U=\prod_{i=1}^{k}\left(\gamma_{\alpha_i}^{U}\right)^{\omega_i},d_1^L=\prod_{i=1}^{k}\left(2-\gamma_{\alpha_i}^{L}\right)^{\omega_i},d_1^U=\prod_{i=1}^{k}\left(2-\gamma_{\alpha_i}^{U}\right)^{\omega_i}$

$a_2^L=\left(1+\mu_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}},a_2^U=\left(1+\mu_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}},b_2^L=\left(1-\mu_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}},b_2^U=\left(1-\mu_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}$

$c_2^L=\left(\gamma_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}},c_2^U=\left(\gamma_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}},d_2^L=\left(2-\gamma_{\alpha_{k+1}}^{L}\right)^{\omega_{k+1}},d_2^U=\left(2-\gamma_{\alpha_{k+1}}^{U}\right)^{\omega_{k+1}}$

$$IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_k\right)=\left\{\left(\left[\frac{a_1^L-b_1^L}{a_1^L+b_1^L},\frac{a_1^U-b_1^U}{a_1^U+b_1^U}\right],\left[\frac{2c_1^L}{d_1^L+c_1^L},\frac{2c_1^U}{d_1^U+c_1^U}\right]\right)\right\}$$

$$\omega_{k+1}\tilde{h}_{k+1}=\left\{\left(\left[\frac{a_2^L-b_2^L}{a_2^L+b_2^L},\frac{a_2^U-b_2^U}{a_2^U+b_2^U}\right],\left[\frac{2c_2^L}{d_2^L+c_2^L},\frac{2c_2^U}{d_2^U+c_2^U}\right]\right)\right\}$$

Hence by Einstein operational law,

$$IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_{k+1}\right)=IVIHFWIA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_k\right)\oplus\omega_{k+1}\tilde{h}_{k+1}$$

$$=\left\{\left(\left[\frac{a_1^L-b_1^L}{a_1^L+b_1^L},\frac{a_1^U-b_1^U}{a_1^U+b_1^U}\right],\left[\frac{2c_1^L}{d_1^L+c_1^L},\frac{2c_1^U}{d_1^U+c_1^U}\right]\right)\right\}\oplus\left\{\left(\left[\frac{a_2^L-b_2^L}{a_2^L+b_2^L},\frac{a_2^U-b_2^U}{a_2^U+b_2^U}\right],\left[\frac{2c_2^L}{d_2^L+c_2^L},\frac{2c_2^U}{d_2^U+c_2^U}\right]\right)\right\}$$

$$= \left\{ \left(\left[\frac{a_1^L a_2^L - b_1^L b_2^L}{a_1^L a_2^L + b_1^L b_2^L}, \frac{a_1^U a_2^U - b_1^U b_2^U}{a_1^U a_2^U + b_1^U b_2^U} \right], \left[\frac{2c_1^L c_2^L}{d_1^L d_2^L + c_1^L c_2^L}, \frac{2c_1^U c_2^U}{d_1^U d_2^U + c_1^U c_2^U} \right] \right) \right\} =$$

$$= \left\{ \left(\left[\frac{\prod_{i=1}^{k+1}\left(1+\mu_{\alpha_i}^L\right)^{\omega_i} - \prod_{i=1}^{k+1}\left(1-\mu_{\alpha_i}^L\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(1+\mu_{\alpha_i}^L\right)^{\omega_i} + \prod_{i=1}^{k+1}\left(1-\mu_{\alpha_i}^L\right)^{\omega_i}}, \frac{\prod_{i=1}^{k+1}\left(1+\mu_{\alpha_i}^U\right)^{\omega_i} - \prod_{i=1}^{k+1}\left(1-\mu_{\alpha_i}^U\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(1+\mu_{\alpha_i}^U\right)^{\omega_i} + \prod_{i=1}^{k+1}\left(1-\mu_{\alpha_i}^U\right)^{\omega_i}} \right], \right. \right.$$

$$\left. \left. \left[\frac{2\prod_{i=1}^{k+1}\left(\gamma_{\alpha_i}^L\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(2-\gamma_{\alpha_i}^L\right)^{\omega_i} + \prod_{i=1}^{k+1}\left(\gamma_{\alpha_i}^L\right)^{\omega_i}}, \frac{2\prod_{i=1}^{k+1}\left(\gamma_{\alpha_i}^U\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(2-\gamma_{\alpha_i}^U\right)^{\omega_i} + \prod_{i=1}^{k+1}\left(\gamma_{\alpha_i}^U\right)^{\omega_i}} \right] \right) \middle/ \alpha_i \in \widetilde{h_i}, (i = 1, 2, ..., k+1) \right\}$$

Thus equation (1) holds for n = k + 1.
Hence by the principle of mathematical induction eqn. (1) is true for all n.
This completes the proof of theorem.

Example: Suppose that

h_1 = {([0.2, 0.3], [0.5, 0.6]), ([0.5, 08], [0.1, 0.2])},

h_2 = {([0.4, 0.6], [0.3, 0.4]), ([0.3, 0.5], [0.1, 0.2])}, and

h_3 = {([0.5, 0.5], [0.2, 0.3]), ([0.2, 0.4], [0.3, 0.6]), ([0.8, 0.9], [0.1, 0.1])}

are three IVIHFEs, and $\omega = (0.6,\ 0.3,\ 0.1)^T$ is their weight vector.

$IVIHFWA_\omega^\varepsilon\left(h_1, h_2, h_3\right) =$

$$\left\{ \begin{array}{l} ([0.295,\ 0.421],[0.326,\ 0.442]),([0.263,\ 0.410],[0.347,\ 0.488]),([0.344,\ 0.494],[0.296,\ 0.438]) \\ ([0.263,\ 0.385],[0.257,\ 0.343]),([0.231,\ 0.374],[0.263,\ 0.381]),([0.313,\ 0.458],[0.246,\ 0.348]), \\ ([0.471,\ 0.727],[0.133,\ 0.206]),([0.444,\ 0.721],[0.135,\ 0.231]),([0.513,\ 0.768],[0.128,\ 0.173]), \\ ([0.444,\ 0.706],[0.083,\ 0.154]),([0.416,\ 0.700],[0.085,\ 0.173]),([0.487,\ 0.749],[0.079,\ 0.129]), \end{array} \right\}$$

Based on the theorem 3.1 we give the following propositions.

Proposition 3.1(Idempotency): Let $h_i = \left\{ \left(\left[\mu_{\alpha_i}^L, \mu_{\alpha_i}^U \right], \left[\gamma_{\alpha_i}^L, \gamma_{\alpha_i}^U \right] \right) / \alpha_i \in h_i \right\}$ $(i = 1, 2, ..., n)$ be a collection of IVIHFSs in L and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of h_i (i=1,2,…,n) such that $\omega_i \in [0,1]$ with $\sum_{i=1}^{n} \omega_i = 1$.

If all h_i (i=1,2,…,n) are equal. i.e., $h_i = h$ (i=1,2,…,n) then $IVIHFWA_\omega^\varepsilon\left(\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_n\right) = \tilde{h}$

Proof: Since $h_i = h$ for all i =1, 2,…, n, $\mu_{\alpha_i}^L = \mu_\alpha^L$, $\mu_{\alpha_i}^U = \mu_\alpha^U$, $\gamma_{\alpha_i}^L = \gamma_\alpha^L$, $\gamma_{\alpha_i}^U = \gamma_\alpha^U$

$$IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_n\right)=\left\{\left(\left[\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{U}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{U}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{U}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{U}\right)^{\omega_i}}\right],\right.\right.$$

$$\left.\left.\left[\frac{2\prod_{i=1}^{n}\left(\gamma_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma_{\alpha_i}^{L}\right)^{\omega_i}},\frac{2\prod_{i=1}^{n}\left(\gamma_{\alpha_i}^{U}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_i}^{U}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma_{\alpha_i}^{U}\right)^{\omega_i}}\right]\right)\middle/\alpha_i\in\widetilde{h_i},(i=1,2,\ldots,n)\right\}$$

$$=IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_n\right)=\left\{\left(\left[\frac{\left(1+\mu_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}-\left(1-\mu_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}}{\left(1+\mu_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}-\left(1-\mu_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}},\frac{\left(1+\mu_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}-\left(1-\mu_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}}{\left(1+\mu_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}-\left(1-\mu_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}}\right],\right.\right.$$

$$\left.\left.\left[\frac{2\left(\gamma_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}}{\left(2-\gamma_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}+\left(\gamma_{\alpha}^{L}\right)^{\sum_{i=1}^{n}\omega_i}},\frac{2\left(\gamma_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}}{\left(2-\gamma_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}+\left(\gamma_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}}\right]\right)\middle/\alpha\in\widetilde{h}\right\}$$

$$=\left\{\left(\left[\frac{\left(1+\mu_{\alpha}^{L}\right)-\left(1-\mu_{\alpha}^{L}\right)}{\left(1+\mu_{\alpha}^{L}\right)-\left(1-\mu_{\alpha}^{L}\right)},\frac{\left(1+\mu_{\alpha}^{U}\right)-\left(1-\mu_{\alpha}^{U}\right)}{\left(1+\mu_{\alpha}^{U}\right)-\left(1-\mu_{\alpha}^{U}\right)}\right],\left[\frac{2\left(\gamma_{\alpha}^{L}\right)}{\left(2-\gamma_{\alpha}^{L}\right)+\left(\gamma_{\alpha}^{L}\right)},\frac{2\left(\gamma_{\alpha}^{U}\right)}{\left(2-\gamma_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}+\left(\gamma_{\alpha}^{U}\right)^{\sum_{i=1}^{n}\omega_i}}\right]\right)\middle/\alpha\in h\right\}$$

$$\left\{\left(\left[\mu_{\alpha}^{L},\mu_{\alpha}^{U}\right],\left[\gamma_{\alpha}^{L},\gamma_{\alpha}^{U}\right]\right)/\alpha\in h\right\}=h$$

Proposition 3.2 (Monotonicity): Let $h_i=\left(\left[\mu_{\alpha_i}^{L},\mu_{\alpha_1}^{U}\right],\left[\gamma_{\alpha_i}^{L},\gamma_{\alpha_i}^{U}\right]\right)$ and h_i^{*} be a collection of IVIHFEs $(i=1,2,\ldots,n)$. If $\mu_{\alpha_i}^{L}\le\mu_{\alpha_i}^{L*},\mu_{\alpha_i}^{U}\le\mu_{\alpha_1}^{U*},\gamma_{\alpha_i}^{L\wedge}\le\gamma_{\alpha_1}^{L\wedge}$ and $\gamma_{\alpha_i}^{U}\le\gamma_{\alpha_i}^{L*}$ for all i, then
$IVIHFWA_{\omega}^{\epsilon}\left(h_1,h_2,\ldots,h_n\right)\le IVIHFWA_{\omega}^{\epsilon}\left(h_1^{*},h_2^{*},\ldots,h_n^{*}\right)$

Proof:

Let $f(x)=\frac{1-x}{1+x},x\in[0,1]$ then it is a decreasing function. If $\mu_{\alpha_i}^{L}\le\mu_{\alpha_i}^{L*}$ for all I, then

$$f(\mu_{\alpha_i}^{L})^{*}\le f(\mu_{\alpha_i}^{L})(\mathrm{i}=1,2,\ldots,\mathrm{n})\ \text{ie}\ \frac{1-\mu_{\alpha_i}^{L*}}{1+\mu_{\alpha_i}^{L*}}\le\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\ \left(i=1,2,\ldots n\right)\ \text{Thus}$$

$$1+\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L*}}{1+\mu_{\alpha_i}^{L*}}\right)^{\omega_i}\le 1+\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\right)^{\omega_i}$$

$$\Leftrightarrow\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\right)^{\omega_i}}\le\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L*}}{1+\mu_{\alpha_i}^{L*}}\right)^{\omega_i}}$$

$$\Leftrightarrow \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\right)^{\omega_i}}-1 \le \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L*}}{1+\mu_{\alpha_i}^{L*}}\right)^{\omega_i}}-1$$

$$\therefore \frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{L}\right)^{\omega_i}} \le \frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{L^*}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{L^*}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}^{L^*}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{L^*}\right)^{\omega_i}} \qquad (6)$$

Let $g(y)=\frac{2-y}{y}, y\in[0,1]$ then g is a decreasing function on $[0,1]$. If $\gamma_{\alpha_i}^{L}\ge\gamma_{\alpha_i}^{L^*}>0$ for all i, then

$g(\gamma_{\alpha_i}^{L^*})\ge g(\gamma_{\alpha_i}^{L})$ (i=1,2,…,n) i.e., $\frac{2-\gamma_{\alpha_i}^{L^*}}{\gamma_{\alpha_i}^{L^*}}\ge\frac{2-\gamma_{\alpha_i}^{L}}{\gamma_{\alpha_i}^{L}}$. Thus

$$\prod_{i=1}^{n}\left(\frac{2-\gamma_{\alpha_i}^{L^*}}{\gamma_{\alpha_i}^{L^*}}\right)^{\omega_i}+1\ge\prod_{i=1}^{n}\left(\frac{2-\gamma_{\alpha_i}^{L}}{\gamma_{\alpha_i}^{L}}\right)^{\omega_i}+1 \Leftrightarrow \frac{2}{\prod_{i=1}^{n}\left(\frac{2-\gamma_{\alpha_i}^{L}}{\gamma_{\alpha_i}^{L}}\right)^{\omega_i}+1}\ge\frac{2}{\prod_{i=1}^{n}\left(\frac{2-\gamma_{\alpha_i}^{L^*}}{\gamma_{\alpha_i}^{L^*}}\right)^{\omega_i}+1}$$

Hence
$$\frac{2\prod_{i=1}^{n}\gamma_{\alpha_i}^{L^{\omega_i}}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\gamma_{\alpha_i}^{L^{\omega_i}}}\ge\frac{2\prod_{i=1}^{n}\gamma_{\alpha_i}^{L^{\omega_i*}}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_i}^{L^*}\right)^{\omega_i}+\prod_{i=1}^{n}\gamma_{\alpha_i}^{L^{*\omega_i}}} \qquad (7)$$

Equation (7) is true when $\gamma_{\alpha_i}^{L}=\gamma_{\alpha_i}^{L^*}=0$ for all i. Based on the definition of score function

$$s\left(IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)\right)\le s\left(IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1^*,\tilde{h}_2^*,...,\tilde{h}_n^*\right)\right)$$

Thus $IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)\le IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1^*,\tilde{h}_2^*,...,\tilde{h}_n^*\right)$.
Hence the proof.

Proposition 3.3 (Boundedness): Let $h_i=\left(\left[\mu_{\alpha_i}^{L},\mu_{\alpha_1}^{U}\right],\left[\gamma_{\alpha_i}^{L},\gamma_{\alpha_i}^{U}\right]\right)$ $(i=1,2,...,n)$ be a collection of IVIHFEs.

Let $\tilde{h}_{\min}=\min_i\left\{\tilde{h}_i\right\}$ and $\tilde{h}_{\max}=\max_i\left\{\tilde{h}_i\right\}$. Then $\tilde{h}_{\min}\le IVIHFWA_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)\le\tilde{h}_{\max}$.

Proof: Let $f(x)=\frac{1-x}{1+x}, x\in[0,1]$. Then f(x) is a decreasing function. Since $\mu_{\alpha_{\min}}^{L}\le\mu_{\alpha_i}^{L}\le\mu_{\alpha_{\max}}^{L}$ for all i, $f\left(\mu_{\alpha_{\max}}^{L}\right)\le f\left(\mu_{\alpha_i}^{L}\right)\le f\left(\mu_{\alpha_{\min}}^{L}\right)$ for all i.

$$\frac{1-\mu_{\alpha_{\max}}^{L}}{1+\mu_{\alpha_{\max}}^{L}}\le\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\le\frac{1-\mu_{\alpha_{\min}}^{L}}{1+\mu_{\alpha_{\min}}^{L}} \qquad (i = 1,2,\ldots,n)$$

$$\Rightarrow\left(\frac{1-\mu_{\alpha_{\max}}^{L}}{1+\mu_{\alpha_{\max}}^{L}}\right)^{\omega_i}\le\left(\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\right)^{\omega_i}\le\left(\frac{1-\mu_{\alpha_{\min}}^{L}}{1+\mu_{\alpha_{\min}}^{L}}\right)^{\omega_i}\Rightarrow\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_{\max}}^{L}}{1+\mu_{\alpha_{\max}}^{L}}\right)^{\omega_i}\le\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_i}^{L}}{1+\mu_{\alpha_i}^{L}}\right)^{\omega_i}\le\prod_{i=1}^{n}\left(\frac{1-\mu_{\alpha_{\min}}^{L}}{1+\mu_{\alpha_{\min}}^{L}}\right)^{\omega_i}$$

$$\Leftrightarrow\left(\frac{1-\mu^L_{\alpha_{\max}}}{1+\mu^L_{\alpha_{\max}}}\right)^{\sum_{i=1}^n\omega_i}\le\prod_{i=1}^n\left(\frac{1-\mu^L_{\alpha_i}}{1+\mu^L_{\alpha_i}}\right)^{\sum_{i=1}^n\omega_i}\le\left(\frac{1-\mu^L_{\alpha_{\min}}}{1+\mu^L_{\alpha_{\min}}}\right)^{\sum_{i=1}^n\omega_i}\Leftrightarrow\frac{1-\mu^L_{\alpha_{\max}}}{1+\mu^L_{\alpha_{\max}}}\le\frac{1-\mu^L_{\alpha_i}}{1+\mu^L_{\alpha_i}}\le\frac{1-\mu^L_{\alpha_{\min}}}{1+\mu^L_{\alpha_{\min}}}$$

$$\Leftrightarrow\left(\frac{2}{1+\mu^L_{\alpha_{\max}}}\right)\le1+\prod_{i=1}^n\left(\frac{1-\mu^L_{\alpha_i}}{1+\mu^L_{\alpha_i}}\right)^{\omega_i}\le\left(\frac{2}{1+\mu^L_{\alpha_{\min}}}\right)\Leftrightarrow\left(\frac{1+\mu^L_{\alpha_{\min}}}{2}\right)\le\frac{1}{1+\prod_{i=1}^n\left(\frac{1-\mu^L_{\alpha_i}}{1+\mu^L_{\alpha_i}}\right)^{\omega_i}}\le\left(\frac{1+\mu^L_{\alpha_{\max}}}{2}\right)$$

$$\Leftrightarrow\mu^L_{\alpha_{\min}}\le\frac{2}{1+\prod_{i=1}^n\left(\frac{1-\mu^L_{\alpha_i}}{1+\mu^L_{\alpha_i}}\right)^{\omega_i}}-1\le\mu^L_{\alpha_{\max}}\Leftrightarrow\mu^L_{\alpha_{\min}}\le\frac{\left(1+\mu^L_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}-\left(1-\mu^L_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}}{\left(1+\mu^L_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}-\left(1-\mu^L_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}}\le\mu^L_{\alpha_{\max}}\ .$$

Similarly $\mu^U_{\alpha_{\min}}\le\dfrac{\left(1+\mu^U_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}-\left(1-\mu^U_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}}{\left(1+\mu^U_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}-\left(1-\mu^U_{\alpha_i}\right)^{\sum_{i=1}^n\omega_i}}\le\mu^U_{\alpha_{\max}}$

Let $g(y)=\dfrac{2-y}{y}, y\in[0,1]$ then g(y) is a decreasing function on $[0,1]$. If $\gamma^L_{\alpha_{\max}}\le\gamma^L_{\alpha_i}\le\gamma^L_{\alpha_{\min}}$ for all i, then $g(\gamma^L_{\alpha_{\min}})\le g(\gamma^L_{\alpha_i})\le g(\gamma^L_{\alpha_{\max}})$ for all i

i.e., $\dfrac{2-\gamma^L_{\alpha_{\min}}}{\gamma^L_{\alpha_{\min}}}\le\dfrac{2-\gamma^L_{\alpha_i}}{\gamma^L_{\alpha_i}}\le\dfrac{2-\gamma^L_{\alpha_{\max}}}{\gamma^L_{\alpha_{\max}}}\Rightarrow\prod_{i=1}^n\left(\dfrac{2-\gamma^L_{\alpha_{\min}}}{\gamma^L_{\alpha_{\min}}}\right)^{\omega_i}\le\prod_{i=1}^n\left(\dfrac{2-\gamma^L_{\alpha_i}}{\gamma^L_{\alpha_i}}\right)^{\omega_i}\le\prod_{i=1}^n\left(\dfrac{2-\gamma^L_{\alpha_{\max}}}{\gamma^L_{\alpha_{\max}}}\right)^{\omega_i}$

$$\Leftrightarrow\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_{\min}}}{\gamma^L_{\alpha_{\min}}}\right)^{\sum_{i=1}^n\omega_i}\le\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_i}}{\gamma^L_{\alpha_i}}\right)^{\sum_{i=1}^n\omega_i}\le\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_{\max}}}{\gamma^L_{\alpha_{\max}}}\right)^{\sum_{i=1}^n\omega_i}$$

$$\Leftrightarrow\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_{\min}}}{\gamma^L_{\alpha_{\min}}}\right)\le\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_i}}{\gamma^L_{\alpha_i}}\right)\le\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_{\max}}}{\gamma^L_{\alpha_{\max}}}\right)\Leftrightarrow\left(\frac{2}{\gamma^L_{\alpha_{\min}}}\right)\le\prod_{i=1}^n\left(\frac{2-\gamma^L_{\alpha_i}}{\gamma^L_{\alpha_i}}\right)^{\omega_i}+1\le\left(\frac{2}{\gamma^L_{\alpha_{\max}}}\right)$$

$\Leftrightarrow\gamma^L_{\alpha_{\max}}\le\dfrac{2}{\prod_{i=1}^n\left(\dfrac{2-\gamma^L_{\alpha_i}}{\gamma^L_{\alpha_i}}\right)^{\omega_i}+1}\le\gamma^L_{\alpha_{\min}}$. This result is true even when L is replaced by U.

Thus $IVIHFWA^{\varepsilon}_{\omega}$ operator is bounded.

4. Interval-valued Intuitionistic Hesitant Fuzzy Einstein Ordered Weighted Averaging ($IVIHFOWA_{\omega}^{\epsilon}$) Operator

In this section $IVIHFOWA_{\omega}^{\epsilon}$ operator is defined and basic properties are stated.

Definition 4.1:

Let $h_i=\left\{\left(\left[\mu_{\alpha_i}^L,\mu_{\alpha_i}^U\right],\left[\gamma_{\alpha_i}^L,\gamma_{\alpha_i}^U\right]\right)/\alpha_i\in h_i\right\}$ $(i=1,2,...,n)$ be a collection of IVIHFSs in L and $\omega=\left(\omega_1,\omega_2,...,\omega_n\right)^T$ is the weight vector of h_i (i=1,2,…,n) such that $\omega_i\in[0,1]$ with $\sum_{i=1}^{n}\omega_i=1$. Then an IVIHFOWA$^\epsilon$ operator of dimension n is a mapping from L^n□□L such that $IVIHFOWA_\omega^\varepsilon\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\omega_1\tilde{h}_{\sigma(1)}\oplus\omega_2\tilde{h}_{\sigma(2)}\oplus...\oplus\omega_n\tilde{h}_{\sigma(n)}$ where (σ(1), σ(2), …, σ(n)) is a permutation of (1, 2,…,n) such that $\tilde{h}_{\sigma(j)}\leq\tilde{h}_{\sigma(j-1)}$ for all j= 2,3,..,n

Theorem 4.1:

Let $h_i=\left\{\left(\left[\mu_{\alpha_i}^L,\mu_{\alpha_i}^U\right],\left[\gamma_{\alpha_i}^L,\gamma_{\alpha_i}^U\right]\right)/\alpha_i\in h_i\right\}$ $(i=1,2,...,n)$ be a collection of IVIHFEs in L. Then their aggregated value by using IVIHFWA$^\epsilon$ operator is also an IVIHFE and

$$IVIHFOWA_\omega^\varepsilon\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\left\{\left(\left[\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}\right],\left[\frac{2\prod_{i=1}^{n}\left(\gamma_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}},\frac{2\prod_{i=1}^{n}\left(\gamma_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}\right]\right)\Big/\alpha_{\sigma(i)}\in\widetilde{h_{\sigma(i)}},(i=1,2,...,n)\right\} \quad (8)$$

where $\omega=\left(\omega_1,\omega_2,...,\omega_n\right)^T$ is the weight vector of h_i (i=1,2,…,n) such that $\omega_i\in[0,1]$ with $\sum_{i=1}^{n}\omega_i=1$ and (σ(1), σ(2), …, σ(n)) is a permutation of (1, 2,…,n) such that $\tilde{h}_{\sigma(j)}\leq\tilde{h}_{\sigma(j-1)}$ for all j= 2,3,..,n.

If $\gamma_{\alpha_i}^L=1-\mu_{\alpha_i}^L,\gamma_{\alpha_i}^U=1-\mu_{\alpha_i}^U$ for all i=1,2,…,n then by equation (8)

$$IVIHFOWA_\omega^\varepsilon\left(h_1,h_2,...,h_n\right)$$

$$=\left\{\left(\left[\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}\right],\left[1-\left(\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^L\right)^{\omega_i}}\right),1-\left(\frac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^U\right)^{\omega_i}}\right)\right]\right)\Big/\alpha_i\in\widetilde{h_i}\right\}$$

Proof: The proof is similar to the theorem 3.1 on $IVIHFWA_{\omega}^{\varepsilon}$ operator.

Example: Suppose that

h_1 = {([0.2, 0.3], [0.5, 0.6]), ([0.5, 08], [0.1, 0.2])}, h_2 = {([0.4, 0.6], [0.3, 0.4]), ([0.3, 0.5], [0.1, 0.2])} and

h_3 = {([0.5, 0.5], [0.2, 0.3]), ([0.2, 0.4], [0.3, 0.6]), ([0.8, 0.9], [0.1, 0.1])} are three IVIHFEs, and $\omega = (0.6,\ 0.3,\ 0.1)^T$ is their weight vector.

$IVIHFOWA_{\omega}^{\varepsilon}(h_1, h_2, h_3)$ =

$$\begin{Bmatrix} ([0.444,\ 0.514],[0.198,\ 0.302]),([0.471,\ 0.570],[0.153,\ 0.254]),([0.416,\ 0.482],[0.129,\ 0.229]) \\ ([0.444,\ 0.540],[0.098,\ 0.192]),([0.263,\ 0.457],[0.261,\ 0.488]),([0.295,\ 0.517],[0.203,\ 0.418]), \\ ([0.231,\ 0.422],[0.171,\ 0.381]),([0.263,\ 0.484],[0.147,\ 0.323]),([0.668,\ 0.808],[0.112,\ 0.148]), \\ ([0.686,\ 0.834],[0.104,\ 0.122]),([0.648,\ 0.793],[0.081,\ 0.110]),([0.668,\ 0.820],[0.061,\ 0.091]), \end{Bmatrix}$$

Proposition 4.1:

(1) Idempotency: Let $h_i = \left\{\left(\left[\mu_{\alpha_i}^L, \mu_{\alpha_i}^U\right], \left[\gamma_{\alpha_i}^L, \gamma_{\alpha_i}^U\right]\right) / \alpha_i \in h_i\right\}$ $(i = 1, 2, ..., n)$ be a collection of IVIHFEs in L

and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of h_i (i=1,2,…,n) such that $\omega_i \in [0,1]$ with $\sum_{i=1}^{n} \omega_i = 1$. If all

$h_i\,(i = 1, 2, ..., n)$ are equal, ie, $h_i = h$ for all i, then $IVIHFOW_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) = h$

(2) Boundedness: Let $h_i = \left(\left[\mu_{\alpha_i}^L, \mu_{\alpha_1}^U\right], \left[\gamma_{\alpha_i}^L, \gamma_{\alpha_i}^U\right]\right)$ $(i = 1, 2, ..., n)$ be a collection of IVIHFEs. Let $\tilde{h}_{\min} = \min_i \{\tilde{h}_i\}$

and $\tilde{h}_{\max} = \max_i \{\tilde{h}_i\}$. Then $h_{\min} \le IVIHFOWA_{\omega}^{\varepsilon}(h_1, h_2, ...h_n) \le h_{\max}$

(3) Monotonicity: Let $h_i = \left\{\left(\left[\mu_{\alpha_i}^L, \mu_{\alpha_i}^U\right], \left[\gamma_{\alpha_i}^L, \gamma_{\alpha_i}^U\right]\right) \big/ \alpha_i \in h_i\right\} (i = 1, 2, ...n)$ be a collection of IVIHFEs. If

$\mu_{\alpha_i}^L \le \mu_{\alpha_i}^{L^*}$ and $\gamma_{\alpha_i}^L \le \gamma_{\alpha_i}^L$ then for all i, then $IVIHFOWA_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \le IVIHFOWA_{\omega}^{\varepsilon}(h_1^*, h_2^*, ..., h_n^*)$

Proof : The proof is similar to that of $IVIHFWA_{\omega}^{\varepsilon}$ operator.

5. Conclusion:

The hesitant fuzzy set (HFS) can deal with the situation where the evaluation of an alternative under each criterion is represented by several possible values, not by a margin of error, or some possibility distribution on the possible values. In this paper, we developed the aggregation operators with interval-valued hesitant fuzzy information based on some Einstein operations. We have also investigated some desirable properties of the proposed interval-valued hesitant fuzzy Einstein aggregation operators, such as Interval valued Intuitionistic Hesitant fuzzy Einstein weighted arithmetic averaging operators ($IVIHFWA^{\varepsilon}_{\omega}$) and interval valued intuitionistic hesitant fuzzy Einstein ordered weighted arithmetic averaging ($IVIHFOWA^{\varepsilon}_{\omega}$) operator. These operators have wide applications in aggregating information in multi criteria decision making problems or in aggregating the judgments of decision makers in group decision making problems.

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