Numerical Approximation of Parabolic Partial Differential Equations with a Retarded Argument

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Abstract

In this paper, an initial boundary value problem with a retarded argument is studied. It is assumed that the coefficients of the given problems are sufficiently smooth. A finite difference scheme is presented in a way that the solution of the problem under consideration converges pointwise at all points of the domain independently of the perturbation parameter and retarded argument. Numerical examples are presented so as to demonstrate the effectiveness of the method.

1. Introduction

In this paper we construct and analyze numerical approximations obtained from finite difference schemes applied to the singularly perturbed initial boundary value problem:

$$L_\varepsilon u(x,t) \equiv pu_t(x,t) - \varepsilon u_{xx}(x,t) + au_x(x,t) + bu(x,t-\tau) = f(x,t) \tag{1.1a}$$

$$u(x,s) = u_0(x,s); \quad x \in \Omega \equiv (0,1), \quad s \in [-\tau, 0], \tag{1.1b}$$

$$u(0,t) = 0, \;\; u(1,t) = 0; \;\; t \in (0,T], \tag{1.1c}$$

where $0 < \epsilon \ll 1$ is a small parameter and $0 \le \tau = o(\epsilon^2)$ is the time delay ($\tau = T/k$ for some integer $k > 1$). Furthermore, we assume that f(x, t), p(x, t), a(x, t) and b(x, t) are sufficiently smooth functions which satisfies,

$$0 < \theta \le b(x,t), \;\; p(x,t) \le \gamma < 0 \;\text{ and }\; a(x,t) \ne 0 \;\text{ for }\; (x,t) \in \Omega \times (0,T],$$

where θ and γ are constants independent of ϵ.

The delay differential is resourceful in mathematical modeling of processes in a variety of application fields. Early use of differential equations with a retarded argument was to describe technical devices. In that context, the delay is a measurable physical quantity. Nowadays these types of problem are ubiquitous in various branches of sciences. In the sciences, stochastic differential equations are used to model systems that are inherently random, or subject to random external perturbations. Furthermore, systems in continuum mechanics or in financial economics have governing equations which involve integral terms representing the effect of the past. During the precedent years numerous approximate methods have been developed and advanced, including the method of averaging, boundary layer method, methods of matched asymptotic expansion [4, 5, 6] and multiple scales [7, 8]. But there are problems associated with these asymptotic methods such as finding the right asymptotic expressions in the inner regions where solution exhibits sharp changes. Instead, research focuses on various particular partial differential equations that are important for applications, within and outside of mathematics. In spite of its great practical value and applications, there does not seem to have been any attempt to study the unsteady parabolic partial differential equations in a general setting with a retarded argument. Our aim in this paper is to carry out such a study. Assuming that the coefficients of the differential equation are smooth, we construct and analyze finite difference method whose solutions converge point-wise at all points of the domain.

2 Auxiliary Results

Let us first setup some notational conventions, we put the usual supremum (semi) norms

$$|v|_{k,D} = \sum_{i+j=k} sup_{(x,t)\in D} |\frac{\partial^{i+j}}{\partial^i x \partial^j t} v(x,t)| \quad \text{and} \quad \|v\|_{k,D} = \sum_{0\leq l\leq k} |v|_{l,D}$$

on $C^k(D)$. For each positive integer k, the space $C^{k,\alpha}(D)$ consists of all function in $C^k(D)$ whose derivatives of order k are Hölder continuous of degree α. For $v \in C^{k,\alpha}(D)$, we define the seminorm

$$\lceil v \rceil_{k,\alpha,D} = \sum_{i+j=k} [\frac{\partial^{i+j}}{\partial^i x \partial^j t} v(x,t)]_{0,\alpha,D}.$$

When the delay values $t-\tau$ are bounded away from t by a positive constant the existence of the solution can be verified using the method of steps. It may be noticed that,

$$u(x,s) = u_0(x,s); \quad x \in \Omega \equiv (0,1), \quad s \in [-\tau, 0),$$
$$u(x,0) = u_0(x); \quad x \in \Omega \equiv (0,1),$$

and therefore $u(x,t-\tau)$ becomes a known function of (x,t) on $[0,1]\times[0,\tau]$. In this case equation (1.1) becomes a classical partial differential equation and can easily be treated using the known existence theories [3]. For a general discussion of the properties enjoyed by the solution of parabolic differential equations we refer to [2]. The following lemma is an immediate consequences of Theorem 3.2 of [1].

Lemma 2.1 *Consider the initial boundary value problem*

$$pu_t(x,t) - \epsilon u_{xx}(x,t) + au_x(x,t) + bu(x,t-\tau) + cu(x,t) = f(x,t) \ \ on \ \ D, \tag{2.1a}$$

$$u = 0 \quad on \ \ \partial D, \tag{2.1b}$$

where a,b, and c are smooth functions on $\overline{D}$. *Let* $\alpha \in (0,1)$.

1. *Suppose that* $f \in C^{0,\alpha}(\overline{D})$. *Then (2.1) has a solution* $u \in C^{1,\alpha}(\overline{D}) \bigcap C^{2,\alpha}(D)$.

2. *Suppose that* $f \in C^{0,\alpha}(\overline{D})$. *Then* $u \in C^{2,\alpha}(\overline{D})$ *if and only if*

$$f(0,0) = f(1,0) = f(1,T) = f(0,T) = 0.$$

If in addition $f \in C^{1,\alpha}(\overline{D})$, *then* $u \in C^{3,\alpha}(\overline{D})$.

Now, we meet the important and recurring concept of an apriori estimates. Since we shall use such apriori estimates later on in several different situations. The differential operator $\mathcal{L}_\varepsilon$ satisfies:

Lemma 2.2 *Suppose that* $\Phi \in C^{2,1}(\overline{D})$ *be such that* $\Phi \geq 0$ *on* ∂D. *If* $L_\epsilon \Phi(x,t) \geq 0$ *for all* $(x,t) \in D$, *then* $\Phi(x,t) \geq 0$ *on* $\overline{D}$.

Proof:- Suppose that $(x^*, t^*) \in \overline{D}$ be such that $\Phi(x^*, t^*) = min_{\overline{D}} \Phi(x,t) \& \ \Phi(x^*, t^*) < 0$. It is then clear from the hypothesis that $(x^*, t^*) \notin \partial D$. Hence $(x^*, t^*) \in D$ and it is easy to follow that $\Phi_x(x^*, t^*) = 0$, $\Phi_t(x^*, t^*) = 0$, and $\Phi_{xx}(x^*, t^*) \geq 0$. Further, $\mathcal{L}_\epsilon \Phi(x^*, t^*) \leq 0$ a contradiction. Therefore, $\Phi(x,t) \geq 0$ for all $x \in \overline{D}$.

□

For obvious reasons the function Ψ is called a barrier function for Φ. We impose the compatibility conditions $u_0(0,0) = u_0(1,0) = 0$, so that the data matches at the corners (0,0) and (1,0) of the domain. As an application of this lemma we immediate obtain Now, we make an attempt to exploit our knowledge of maximum principal to obtain a priori bounds on the exact solution and its derivatives. It is easy to obtain;

Theorem 2.3 *There exist a number* C *independent of the* ϵ *such that for all sufficiently small positive values of* ϵ *the relation*

$$\begin{aligned} \|u(x,t) - \varphi(x)\| &\leq CT, \quad and \\ \|u(x,t)\| &\leq C, \quad t \in (0,T]. \end{aligned} \tag{2.2}$$

Suppose that the compatibility condition of lemma (2.1) holds so that the solution of the boundary-value problem is smooth enough for each but fixed value of the parameter. At corners assume that

$$\frac{\partial^k \varphi}{\partial x^k} = 0, \quad \frac{\partial^{k_0} \Phi}{\partial t^{k_0}} = 0, \quad \frac{\partial^{k_0} \Psi}{\partial t^{k_0}} = 0; \quad k + 2k_0 \leq [\nu] + 2n,$$

$$\frac{\partial^{k+k_0} f(x,t)}{\partial x^k \partial t^{k_0}} = 0; \quad k + 2k_0 \leq [\nu] + 2n - 2$$

where $[\nu]$ is the integer part of the number ν, $\nu > 0$, $n \geq 0$ is an integer and $[\nu] + 2n \geq 0$. Using interior a priori estimates and estimates up to the boundary for the regular function $\breve{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\epsilon$, we find the following estimate for $(x,t) \in \overline{D}$

$$\| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t) \| \leq C/\varepsilon^k, \quad k + 2k_0 \leq 2n + 4, \quad n \geq 0.$$

3 Solution Methodology

Consider the uniform time grid

$$T_j = \{t_j : t_j = j\frac{T}{M} = j\Delta t,\ j = 0, 1, \ldots M\}.$$

Discretization of time variable by mean of the Euler's implicit rule yields

$$L_{x,\epsilon}u_{j+1} = \Delta t\, f(x, t_{j+1}) + p_{j+1}(x)u_j - b_{j+1}(x)\Delta t\, u_{j-k+1} := h(x), \tag{3.1a}$$

$$u(x, 0) = u_0; \quad x \in \Omega \equiv (0, 1), \tag{3.1b}$$

$$u_{j+1}(0, t) = u_{j+1}(1, t) = 0, \quad \text{for } t \in (0, T], \ \ j = 0, \ldots, M-1. \tag{3.1c}$$

This gives semidiscrete approximation $u_j(x)$ to the exact solution $u(x,t)$ of (1.1) at all the time levels given by $t_j = j\Delta t$. Clearly, the semidiscrete operator $L_{x,\epsilon}$ satisfies maximum principle and consequently

$$\|L^{-1}_{x,\epsilon}\|_\infty \le C'. \tag{3.2}$$

This ensures the stability of the method (3.1). Moreover, the following accuracy results holds:

Theorem 3.1 *If $|\partial^i u(x,t)/\partial t^i| \le C$ for $i \in \{0, 1, 2\}$ and $(x,t) \in \overline{\Omega} \times (0, T]$. Then, the local and global truncation error, respectively, satisfies the following estimates:*

$$\|e_{j+1}\| = \|u(t_{j+1}) - \tilde{u}_{j+1}\| \le C\Delta t^2, \tag{3.3}$$

$$\|E_j\| = \|u(x, t_j) - u_j(x)\| \le C\Delta t. \tag{3.4}$$

Proof:- Since the function $\tilde{u}_{j+1}$ satisfies maximum principle and the solution of the problem (1.1) being smooth enough, it holds

$$\begin{aligned} u(t_j) &= u(t_{j+1} - \Delta t) \\ &= u(t_{j+1}) - \Delta t\, \frac{\partial u}{\partial t}(t_{j+1}) + \int_{t_j}^{t_{j+1}} (t_j - \xi)\frac{\partial^2 u}{\partial t^2}(\xi)\, d\xi. \end{aligned}$$

Note that the local truncation error e_{j+1} is the solution of boundary value problem of type

$$L_{x,\epsilon}e_{j+1} = O(\Delta t^2); \quad e_{j+1}(0) = e_{j+1}(1) = 0.$$

An application of stability result (3.2) immediately leads to (3.3). Moreover,

$$\|E_j\| = \|\sum_{k=1}^{j} e_k\| = \|e_1\| + \|e_2\| + \ldots + \|e_j\| \le (CT)\Delta t = C\Delta t,$$

since $j\Delta t \le T$. Therefore, the time semidiscretization process is uniformly convergent of first order. As an application of the maximum principal we can immediately obtain the bounds on the solution of the semidiscrete problem class.

Lemma 3.2 *Let u^ϵ_{j+1} be the solution of a problem from semidiscrete problem class (3.1), then*

$$\|u^\epsilon_{j+1}\| \le \|p_{j+1}\|\|u^\epsilon_j\| + \Delta t(\|b_{j+1}\|\|u^\epsilon_{j-k+1}\| + \|f_{j+1}\|). \tag{3.5}$$

Moreover, for k=1,2,3

$$\|(u^\epsilon_{j+1})^k\| \le C\epsilon^{-k}. \tag{3.6}$$

Proof:- Let us construct two barrier function $\zeta^\pm$ defined by

$$\zeta^\pm(x) = \|p_{j+1}\|\|u^\epsilon_j\| + \Delta t(\|b_{j+1}\|\|u^\epsilon_{j-k+1}\| + \|f_{j+1}\|) \pm u^\epsilon_{j+1}(x).$$

Note that $\zeta^\pm \ge 0$, for $x \in \{0,1\}$. Further,

$$L_{x,\epsilon}\zeta^\pm(x) \;=\; \|p_{j+1}\|\|u^\epsilon_j\| + \Delta t(\|b_{j+1}\|\|u^\epsilon_{j-k+1}\| + \|f_{j+1}\|) \pm L_{x,\epsilon}u^\epsilon_{j+1}(x) \ge 0$$

for all x. An application of maximum principle asserts that $\zeta^\pm(x) \ge 0$ for all $x \in \overline{\Omega}$ and consequently (3.5) holds.

Now to find out the bounds on the derivatives of the solution of (3.1), let $x \in \Omega$ and construct a neighborhood $N_x = (c, c+\epsilon\Delta t)$, where c is a positive constant chosen so that $x \in N_x \subset \Omega$. Then by the Mean Value Theorem, there exists a point $x^* \in N_x$, such that

$$\epsilon\Delta t|(u^\epsilon_{j+1})'(x^*)| \le 2\|u^\epsilon_{j+1}\|. \tag{3.7}$$

Integration of (3.1) over the line segment x^* to x yields

$$\begin{aligned}\epsilon\Delta t(u^\epsilon_{j+1})_x(x) - \epsilon\Delta t(u^\epsilon_{j+1})_x(x^*) \;=\; & \int_{x^*}^{x}[p_{j+1}(s)u_j(s) - b_{j+1}(s)\Delta t\; u_{j-k+1}(s) + \Delta t\; f_{j+1}(s) \\ & -a_{j+1}(s)\Delta t(u^\epsilon_{j+1})_x(s) - p_{j+1}(s)u^\epsilon_{j+1}(s)]ds. \end{aligned} \tag{3.8}$$

Taking modulus on both the sides and using the fact that the maximum norm of a function is always greater than the value of the function itself over the domain of consideration, we get

$$\begin{aligned}\epsilon\Delta t|(u^\epsilon_{j+1})_x(x)| \;\le\; & \epsilon\Delta t|(u^\epsilon_{j+1})_x(x^*)| + (\|p_{j+1}\|(\|u_j\| + \|u_{j+1}\|) + \Delta t\|b_{j+1}\|\|u_{j-k+1}\| \\ & +\Delta t\|f_{j+1}\|)|(x - x^*)| + (2\|a_{j+1}\| + \|a'_{j+1}\|\|u^\epsilon_{j+1}\|)\Delta t. \end{aligned} \tag{3.9}$$

An application of Lemma 3.2 together with the inequalities (3.7), (3.8) yields the required result. We now present the totally discrete scheme obtained after the spatial discretization of (3.1). Let us introduce a uniform grid

$$X_i = \{x_i : x_i = i\frac{1}{N} = ih,\; i = 0, 1, \ldots N\}.$$

At each time level $j+1$, discretization of (3.1) by means of simple upwind finite difference operators on X_i, yields

$$L^h_{x,\epsilon} = E_i u_{i-1} - F_i u_i + G_i u_{i+1} = H_i, \tag{3.10a}$$

$$u(0) = 0, \;\; u(1) = 0, \tag{3.10b}$$

where

$$E_i = \frac{-\varepsilon \Delta t}{h^2}, \quad F_i = \frac{-2\varepsilon \Delta t}{h^2} + \frac{a_i \Delta t}{h} - p_i, \quad G_i = \frac{-\varepsilon \Delta t}{h^2} + \frac{a_i \Delta t}{h},$$

$$H_i = \Delta t\, f(x_i, t_{j+1}) + p_i(t_{j+1}) u_i(t_j) - b_i(t_{j+1}) \Delta t\, u_i(t_{j-k+1}), \text{ for } i = 1, 2, \ldots N-1.$$

The difference equations (3.10), at each time level $j+1$, form a tridiagonal system of $N-1$ equations with $N+1$ unknowns. The coefficient matrix of such a system of equations is nonsingular if it is either strictly diagonally dominant or irreducible diagonally dominant [?]. To solve this system of difference equations, we will use the discrete invariant imbedding algorithm.

4 Numerical Illustration

In this section we present numerical results of two examples which support the theory and demonstrate the potential of our approach.

Example 4.1 *Consider the problem (1.1) on the unit square* $(0,1) \times (0,1]$ *with* $a = 1$, $b = 1$, $u_0 = 0$ *and right hand side is given by*

$$f(x,t) = x\tau(-1 + t - \tau)\left(1 - \exp\left(-\frac{1-x}{\epsilon}\right)\right) + t\tau\left(1 + \exp\left(-\frac{1-x}{\epsilon}\right)\right).$$

Table 1: The maximum absolute error for example (4.1) for $\tau = 0.5$.

$\varepsilon \downarrow \quad N \rightarrow$	64	128	256	512	1024
2^{-1}	0.00177172	0.00088967	0.00044580	0.00022314	0.00011163
2^{-2}	0.00550782	0.00278395	0.00139980	0.00070185	0.00035141
2^{-3}	0.01252152	0.00640446	0.00324035	0.00162962	0.00081721
2^{-4}	0.02516028	0.01316166	0.00673881	0.00341094	0.00171605
2^{-5}	0.04732857	0.02575017	0.01348362	0.00690729	0.00349688
2^{-6}	0.08267355	0.04785269	0.02605624	0.01365001	0.00699419

Table 2: Maximum absolute error for different values of τ where $\epsilon = 1$.

$\delta \downarrow \quad N \rightarrow$	64	128	256	512	1024
Example (4.1)					
0.0	0.00208603	0.00053962	0.00013610	0.00003410	0.00000853
0.1	0.00208637	0.00053952	0.00013607	0.00003409	0.00000853
0.2	0.00208663	0.00053940	0.00013603	0.00003408	0.00000853
0.3	0.00208680	0.00053928	0.00013599	0.00003407	0.00000852

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