Inclusion Theorem on Two Summability Methods

P. L. Powar 1[#1], Yuwraj Kumar Tiwari 2[#2]

#1 Department of Mathematics and Computer Science , R.D.V.V., Jabalpur.
#2 Department of Mathematics and Computer Science , R.D.V.V., Jabalpur.

ABSTRACT

In this paper, the authors have defined $|\overline{N}, p_n, \lambda_n|_k$ $(k \geq 1)$ summability and established that $|\overline{N}, p_n, \lambda_n|_k$ method is included in $|C, 1|_k$ method.

Key words: Summability, inclusion and absolute summability.

Corresponding Author: **P. L. POWAR**

INTRODUCTION

Given $\sum a_n$, let (s_n) be its sequence of partial sums and $u_n = na_n$. The n[th] *Cesàro* means of order $\alpha(\alpha > -1)$ of the sequence (s_n) and (u_n) are denoted by s_n^α and t_n^α respectively. The series $\sum a_n$ is said to be absolutely summable (C, α) with index k, or simply summable $|C, \alpha|_k$ $(k \geq 1)$ (cf. [4]), if

$$\sum_{n=1}^{\infty} n^{k-1} |s_n^\alpha - s_{n-1}^\alpha|^k < \infty \tag{1.1}$$

Since t_n^α = n$(s_n^\alpha - s_{n-1}^\alpha)$ (cf. [5]), condition (1.1) reduces to

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty \tag{1.2}$$

Consider a sequence (pn) of positive real numbers such that

$$Pn = \sum_{v=0}^{n} p_v \to \infty as n \to \infty \quad (P_{-1} = p_{-1} = 0) \tag{1.3}$$

Let (λn) be the sequence of positive real numbers such that

- (λ_n) is decreasing
- $\lambda_n \to 0 \ as n \to \infty$

The transformed sequence (T_n) of the $(\overline{N}, p_n,)$ mean of sequence (s_n), generated by a sequence of coefficients (p_n) is defined as (cf. [3]),

$$Tn = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, k≥1 (cf. [1]), if

$$\sum_{n=1}^{\infty} [\frac{P_n}{p_n}]^{k-1} |T_n - T_{n-1}|^k < \infty \tag{1.5}$$

In the present paper, the sequence-to-sequence transformation $(T_n^\star)$ of the $(\overline{N}, p_n, \lambda_n)$ mean of the sequence (s_n) generated by the sequence of coefficient (p_n) and the sequence (λ_n) is defined as

$$T_n^\star = \frac{1}{P_n \lambda_n} \sum_{v=0}^{n} p_v s_v \tag{1.6}$$

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n, \lambda_n|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty}(\lambda_n)^k \left[\frac{P_n}{p_n}\right]^{k-1} \ |T_n^\star - T_{n-1}^\star|^k < \infty \tag{1.7}$$

RESULTS ALREADY PROVED

Bor [3] (see [2] also) has established following two interesting results and our result is based on these Theorems:

Theorem 2.1 [2] Let (p_n) be a sequence of positive real constants such that as n→ ∞

$$\text{(i)} \quad np_n = O(P_n) \qquad \text{(ii)} \ P_n = O(np_n) \tag{2.1}$$

If $\sum a_n$ is summable $|C,1|_k$, then it is also summable $|N,p_n|_k$, $k \geq 1$.

Theorem 2.2 [3] Let (p_n) be a sequence of positive real constants such that it satisfies the condition (2.1). If $\sum a_n$ is summable $|N,p_n|_k$, then itis also summable $|C,1|_k$, $k \geq 1$.

MAIN RESULT

We now state our main Theorem which is similar to Theorem 2.2.

Theorem 3.1 Let (p_n) and (λ_n) be the sequence of positive real numbers such that

(i) $P_n = O(np_n), P_n = \sum_{v=0}^{n} p_v \to \infty$ as $n \to \infty$.

(ii) $\lambda_n \to 0 \ as \ n \to \infty$ and $\lambda_n/\lambda_{n-1} < P_{n-1}/P_n$

If $\sum a_n$ is summable $|\overline{N},p_n,\lambda_n|_k$ then it is also summable $|C,1|_k, k \geq 1$.

Proof of Theorem 3.1 We denote the $n^{th}(C,1)$ mean of the sequence(na_n) by

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v \tag{3.1}$$

Given, the series $\sum a_n$summable by $|\overline{N},p_n,\lambda_n|_k k \geq 1$, means that

$$\sum_{n=1}^{\infty}(\lambda_n)^k \left[\frac{P_n}{p_n}\right]^{k-1} |T_n^\star - T_{n-1}^\star|^k < \infty \tag{3.2}$$

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} v(s_v - s_{v-1})$$

Breaking the above sum into two parts, applying the change of variable and writing two sums together, we get

$$t_n = \frac{-\sum_{v=0}^{n} s_v}{n+1} + s_n (3.3)$$

Consider $$T_n^\star = \frac{1}{P_n\lambda_n}\sum_{v=0}^{n} p_v s_v \tag{3.4}$$

Using (3.4), we obtain

$$s_n = \frac{1}{p_n}[P_n T_n^{\star}\lambda_n - P_{n-1}T_{n-1}^{\star}\lambda_n] \tag{3.5}$$

Substituting the values of sn and s_v from (3.5) in (3.3), we get

$$t_n = -\frac{1}{n+1}[\sum_{v=0}^{n}\frac{1}{p_v}\{P_v T_v^{\star}\lambda_v - P_{v-1}T_{v-1}^{\star}\lambda_v\}] + \frac{1}{p_n}[P_n T_n^{\star}\lambda_n - P_{n-1}T_{n-1}^{\star}\lambda_{n-1}]$$

$$= -\frac{1}{n+1}\left[\sum_{v=0}^{n}\frac{P_v T_v^{\star}\lambda_v}{p_v} - \sum_{v=-1}^{n-1}\frac{P_v T_v^{\star}\lambda_v}{p_{v+1}}\right] + \frac{1}{p_n}[P_n T_n^{\star}\lambda_n - P_{n-1}T_{n-1}^{\star}\lambda_{n-1}]$$

Since, $\frac{1}{p_{v+1}} < \frac{1}{p_v}$ and $P_{-1} = 0$, we get

$$t_n \leq -\frac{1}{n+1}[\frac{P_n T_n^{*}\lambda_n}{p_n}] + \frac{1}{p_n}[P_n T_n^{\star}\lambda_n - P_{n-1}T_{n-1}^{\star}\lambda_{n-1}]$$

$$\leq \frac{P_n T_n^{*}\lambda_n}{p_n}[\frac{n}{n+1}] - \frac{P_{n-1}T_{n-1}^{*}\lambda_{n-1}}{p_n}$$

$$\leq \frac{P_n T_n^{*}\lambda_n}{p_n} - \frac{P_{n-1}T_{n-1}^{*}\lambda_{n-1}}{p_n}$$

In view of condition (ii), we obtain

$$t_n \leq \{\lambda n \left[\frac{P_n}{p_n}\right][T_n^{*} - T_{n-1}^{*}]\}$$

Now, applying Hölder's inequality, we get

$$\sum_{n=1}^{\infty}\frac{1}{n}|t_n|^k \leq \sum_{n=1}^{\infty}\frac{1}{n}(\lambda_n)^k\left[\frac{P_n}{p_n}\right]^k |T_n^{*} - T_{n-1}^{*}|^k$$

In view of condition (i), we know $\frac{1}{n} = O(\frac{p_n}{P_n})$, hence we get

$$\sum_{n=1}^{\infty}\frac{1}{n}|t_n|^k \leq \sum_{n=1}^{\infty}(\lambda_n)^k\left[\frac{P_n}{p_n}\right]^{k-1} |T_n^{*} - T_{n-1}^{*}|^k$$

Using (3.2), we finally conclude that

$$\sum_{n=1}^{\infty}|t_n|^k < \infty$$

Thus, $|\bar{N}, p_n, \lambda_n|_k \subseteq |C, 1|_k$

CONCLUSION

Besides Theorem 2.2 of [3], our result covers all general cases for different choices of the sequence (λn) except λn = 1, for all n.

REFERENCE

[1]. Bor, H., On $|\bar{N}, p_n|_k$ summability factors of infinite series, J. Univ. Kuwait, Sci., 10, 37-42, 1983.

[2]. Bor, H., On two summability methods, Math. Proc. Cambridge Philos. Soc., 97, 147-149, 1985.
[3]. Bor, H., A note on two summability methods, Proceedings of the American Mathematical Society, Vol. 98, No. 1, 81-84, 1986.
[4]. Flett, T. M., On an extension absolute summability and theorems of Littlewood and Paley, Proc. London Math. Soc., 7, 113-141, 1957.
[5]. Kogbetliantz, E., Sur les series absolument sommables par la methode des moyennes arithmetiques, Bull. Sci. Math., 9, 234-256, 1925.