COUPLED FIXED POINT THEOREMS IN S-METRIC SPACES WITH MIXED G-MONOTONE PROPERTY

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Abstract. In this manuscript we prove some coupled fixed point theorems in S-metric space susing the mixed g-monotone property. We give some examples in support of our results.

1. Introduction

The advancement and the rich growth of fixed point theorems in metric spaces have important theoretical and practical applications. It has remarkable influence on applications such as the theory of differential and integral equations [1]. Metric spaces have very wide applications in mathematics and applied sciences. For this many authors tried to give definitions of metric spaces in many ways. In 1989, Gahler [4, 5], introduced the notion of 2-metric spaces and Dhage [3] introduced the notion of D-metric spaces. After the introduction of these metric spaces many authors proved some fixed point results related to these metric spaces. After this Mustafa and Sims [2] proved that most of the results of Dhage's D-metric spaces are not valid. So, they introduced the new concept of generalized metric space called G-metric space and give some remarkable results in G-metric spaces. Now, recently Sedghi et al. [6] have introduced the notion of S-metric spaces as the generalization of G-metric and D*-metric spaces. Some results have been obtained in [6, 7, 8] by Sedghi et al. In this paper, we prove some coupled coincidence point results in S-metric space using the mixed g-monotone property which are the generalizations of some fixed point theorems in metric spaces [9, 10, 11, 12, 13].

2. Preliminaries

Here we give some definitions which are throughout used in this paper.

Definition 2.1 ([6])**.** Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$.

(i) $S(x, y, z) \geq 0$
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

Then the pair (X, S) is called an S-metric space.

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Definition 2.2 ([15]). Let $(X, \leq)$ be a partially ordered set equipped with a metric S such that (X, S) is a metric space. Further, equip the product space $X \times X$ with the following partial ordering:

$$\text{for } (x, y), (u, v) \in X \times X,$$
$$\text{define } (u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v.$$

Definition 2.3 ([15]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. One says that F enjoys the mixed monotone property if (x, y) is monotonically nondecreasing in x and monotonically nonincreasing in y; that is, for any $x, y \in X$,

$$x^1, x^2 \in X,\ x^1 \leq x^2 \quad \Rightarrow \quad F(x^1, y) \leq F(x^2, y),$$
$$y^1, y^2 \in X,\ y^1 \leq y^2 \quad \Rightarrow \quad F(x, y^1) \geq F(x, y^2).$$

Definition 2.4 ([15]). An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

Lemma 2.5 ([8]). *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Definition 2.6 ([14]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is,

if, for all $x^1, x^2 \in X$, $g(x^1) \leq g(x^2)$ implies $F(x^1, y) \leq F(x^2, y)$, for any $y \in X$, and,

for all $y^1, y^2 \in X$, $g(y^1) \leq g(y^2)$ implies $F(x, y^1) \geq F(x, y^2)$, for any $x \in X$.

Definition 2.7 ([14]). An element $(x, y) : X \times X$ is called a coupled coincidence point of mappings $: X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \quad F(y, x) = g(y).$$

3. Main Results

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and assume that there is a metric S on X such that (X, S) is a complete S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi[S(F(x, y), F(x, y), F(u, v)] \leq \frac{1}{2}\varphi[S(gx, gx, gu) + S(gy, gy, gv)] - \phi[S(gx, gx, gu) + S(gy, gy, gu)]$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gv \geq gy$.

Suppose that $F(X \times X) \subset g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that then there exist $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$.

Then there exist $x, y \in X$ such that

$$F(x, y) = gx \quad \text{and} \quad F(y, x) = gy$$

that is, F and g have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \text{and} \quad gy_0 \geq F(y_0, x_0).$$

But $F(X \times X) \subset g(X)$, so, we can take $x_1, y_1 \in X$ such that

$$gx_1 \leq F(x_0, y_0) \quad \text{and} \quad gy_1 \geq F(y_0, x_0). \tag{1}$$

Taking $F(X \times X) \subset g(X)$, by continuous this process, we can take sequences x_n and y_n in X such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n). \tag{2}$$

We shall prove that

$$gx_n \leq gx_{n+1} \quad \text{and} \quad gy_{n+1} \geq gy_n \quad \text{for } n = 0, 1, 2, 3, ... \tag{3}$$

For this, we use mathematical induction. Since $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. Then by equation (2), we obtain

$$gx_0 \leq gx_n \quad \text{and} \quad gy_n \geq gy_0 \quad \text{for } n = 0, 1, 2, 3, ... \tag{4}$$

i.e. (4) holds for $n = 0$. We suppose that equation (4) holds for some $n > 0$. As F has the mixed g-monotone property and $gx_n \leq gx_{n+1}$ and $gy_{n+1} \geq gy_n$, we get

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n) \leq F(x_{n+1}, y_n) \\ &\leq F(x_{n+1}, y_{n+1}) \\ &= gx_{n+2}, \\ gy_{n+2} &= F(y_{n+1}, x_{n+1}) \leq F(y_{n+1}, x_n) \\ &\leq F(y_n, x_n) \\ &= gx_{n+1} \end{aligned}$$

Thus equation (4) holds for any $n \in \mathbb{N}$. Suppose, for some $n \in \mathbb{N}$, that

$$gx_n = gx_{n+1} \quad \text{and} \quad gy_n = gy_{n+1}$$

then, by equation (3) (x_n, y_n) is a coupled coincidence point of F and g. From now on, suppose that for any $n \in \mathbb{N}$ that atleast $gx_n \neq gx_{n+1}$ and $gy_n \neq gy_{n+1}$.

By equations (1)-(4), we get

$$\begin{aligned} &\psi(S(gx_{n+1}, gx_{n+1}, gx_{n+2})) \\ &\quad = \psi(S(F(x_n, y_n), F(x_n, y_n), F(x_{n+1}, y_{n+1}))) \\ &\quad \leq \frac{1}{2}\psi[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})] \\ &\qquad\qquad - \phi[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})] \end{aligned} \tag{5}$$

$$\begin{aligned} &\psi(S(gy_{n+1}, gy_{n+1}, gy_{n+2})) \\ &\quad = \psi(S(F(y_n, x_n), F(y_n, x_n), F(y_{n+1}, x_{n+1}))) \\ &\quad \leq \frac{1}{2}\psi[S(gy_n, gy_n, gy_{n+1}) + S(gx_n, gx_n, gx_{n+1})] \\ &\qquad\qquad - \phi[S(gy_n, gy_n, gy_{n+1}) + S(gx_n, gx_n, gx_{n+1})] \end{aligned} \tag{6}$$

From equation (5) and equation (6), we obtain that

$$\begin{aligned}&\psi(S(gx_{n+1}, gx_{n+1}, gx_{n+2})) + \psi(S(F(gy_{n+1}, gy_{n+1}, gy_{n+2})))\\&\quad \le \psi[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})]\\&\qquad - 2\phi[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})] \end{aligned} \tag{7}$$

By the property of (iii) of ψ, we get

$$\begin{aligned}&\psi[S(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S(gy_{n+1}, gy_{n+1}, gy_{n+2})]\\&\quad \le \psi[S(gx_{n+1}, gx_{n+1}, gx_{n+2})] + \psi[S(gy_{n+1}, gy_{n+1}, gy_{n+2})]\end{aligned} \tag{8}$$

Combining (7) and (8), we have that

$$\begin{aligned}&\psi[S(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S(gy_{n+1}, gy_{n+1}, gy_{n+2})]\\&\quad \le \psi[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})]\\&\qquad - 2\phi[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})]\end{aligned} \tag{9}$$

Let

$$\delta_n = S(gx_n, gx_n, gx_{n-1}) + S(gy_n, gy_n, gy_{n-1})\,. \tag{10}$$

Then, we get

$$\psi(\delta_{n+2}) \le \psi(\delta_{n+1}) - 2\psi(\delta_{n+1}),\ \text{for all } n$$

which gives that

$$\psi(\delta_{n+2}) \le \psi(\delta_{n+1}),\ \text{for all } n.$$

Since ψ is nondecreasing, we have that $\delta_{n+2} \le \delta_{n+1}$ for all n. Thus $\{\delta_n\}$ is a nonincreasing sequence. But it is bounded below from 0, there is some $\delta \ge 0$ such that

$$\lim_{t\to 0^+} \delta_n = \delta\,. \tag{11}$$

We shall prove that $\delta = 0$. Assume, on the contrary that $\delta > 0$. Letting $n \to \infty$ in (10) and having in mind that we suppose that $\lim_{t\to r} \phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \phi(t) = 0$, we get

$$\psi(\delta) \le \psi(\delta) - 2\phi(\delta) < \psi(\delta) \tag{12}$$

which gives us a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [S(gx_n, gx_n, gx_{n-1}) + S(gy_n, gy_n, gy_{n-1})] = 0\,. \tag{13}$$

Now, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the metric space (X, S). Suppose, on the contrary, that, one of the sequences $\{gx_n\}$ and gy_n is not a Cauchy sequence.

That is,

$$\lim_{n,m\to\infty} S(gx_m, gx_m, gx_n) \ne 0$$

or

$$\lim_{n,m\to\infty} S(gy_m, gy_m, gy_n) \ne 0.$$

This means that there exists an $\epsilon > 0$, for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of x_n and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of y_n with $n(k) \geq m(k) \geq k$ such that

$$S(gx_{n(k)}, gx_{m(k)}, gx_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) \geq \epsilon\,. \tag{14}$$

Now, by virtue of $m(k)$, we can take $n(k)$ is such a way that it is the smallest integer with $n(k) > m(k) \geq k$ satisfying (14). We have

$$S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)-1}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)-1}) < \epsilon\,. \tag{15}$$

Now, using triangle inequality, we get

$$\begin{aligned} S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}) &= S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \\ &\leq S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) \\ &\quad + S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) \\ &\quad + S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)-1}) \end{aligned} \tag{16}$$

and

$$\begin{aligned} S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) &= S(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\ &\leq S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \\ &\quad + S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \\ &\quad + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)-1}) \end{aligned} \tag{17}$$

Adding (16) and (17) and using equation (14) and (15), we get

$$\begin{aligned} \epsilon &\leq S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) \\ &\leq S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) \\ &\quad + S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)-1}) + S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \\ &\quad + S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)-1}) \\ &< \epsilon + 2S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + 2S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \end{aligned}$$

Letting $k \to \infty$ and having in mind equation (13), we obtain

$$\begin{aligned} \lim_{k\to\infty} \lambda_k &= \lim_{k\to\infty} [S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)})] \\ &= \epsilon \end{aligned}$$

Again, using the triangle inequalities, we get

$$\begin{aligned} \lambda_k &= S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) \\ &\leq S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) \\ &\quad + S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + S(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) \\ &\quad + S(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) + S(gy_{n(k)}, gy_{n(k)}, gy_{m(k)+1}) \\ &\leq 2S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{m(k)+1}) \\ &\quad + S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{n(k)+1}) + S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)+1}) \end{aligned}$$

$$\begin{aligned} &+ 2S(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) + S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)+1}) \\ &+ S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{n(k)+1}) + S(gy_{n(k)}, gy_{n(k)}, gy_{m(k)+1}) \\ \leq\ & 2\delta_{m(k)+1} + \delta_{n(k)+1} + 2S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{n(k)+1}) \\ &+ 2S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)+1}) \end{aligned} \tag{18}$$

Since $n(k) \geq m(k)$, so

$$gx_{m(k)} \leq gx_{n(k)} \quad \text{and} \quad gy_{m(k)} \geq gy_{n(k)}\,. \tag{19}$$

Thus by equation (1), (3) and (18) we have that

$$\begin{aligned} &\psi(S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{n(k)+1})) \\ &\quad = \psi[S(F(x_{m(k)}, y_{m(k)}), F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)}))] \\ &\quad \leq \frac{1}{2}\psi[S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)})] \\ &\qquad - \phi[S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)})] \end{aligned} \tag{20}$$

$$\begin{aligned} &\psi(S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{n(k)+1})) \\ &\quad = \psi[S(F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)}))] \\ &\quad \leq \frac{1}{2}\psi[S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) + S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)})] \\ &\qquad - \phi[S(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}) + S(gy_{m(k)}, gy_{m(k)}, gy_{n(k)})] \end{aligned} \tag{21}$$

Now, combining (18), (20) and (21), we get

$$\begin{aligned} \psi(\lambda_k) &\leq \psi[2\delta_{m(k)+1} + \delta_{n(k)+1} + 2S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{n(k)+1})) \\ &\qquad + 2S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{n(k)+1})] \\ &\leq \psi(2\delta_{m(k)+1} + \delta_{n(k)+1}) + \psi(2S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{n(k)+1}) \\ &\qquad + 2S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{n(k)+1}))] \end{aligned}$$

$$\begin{aligned} &\leq \psi(2\delta_{m(k)+1}) + \psi(\delta_{n(k)+1}) + \psi(2S(gx_{m(k)+1}, gx_{m(k)+1}, gx_{n(k)+1})) \\ &\quad + \psi((2S(gy_{m(k)+1}, gy_{m(k)+1}, gy_{n(k)+1}))] \\ &\leq \psi(2\delta_{m(k)+1}) + \psi(\delta_{n(k)+1}) + \psi(\lambda_k) - 2\phi(\lambda_k) \end{aligned}$$

Now, assuming $k \to \infty$,we obtain a contradiction. This gives that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the metric space (X, S). But, we have that (X, S) is complete, so there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} gy_n = y\,. \tag{22}$$

Again from equation (22) and using the continuity of the function g, we get

$$\lim_{n\to\infty} g(gx_n) = gx \quad \text{and} \quad \lim_{n\to\infty} g(gy_n) = gy\,. \tag{23}$$

It gives from equation (3) and the continuity of F and the function g that

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n) \tag{24}$$

and

$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n) \tag{25}$$

Now, we shall show that $gx = F(x, y)$ and $gy = F(y, x)$. By assuming $n \to \infty$ in (24) and (25), by (22), (23) and using the continuity of F, we get

$$\begin{aligned} gx &= \lim_{n\to\infty} g(gx_n) \\ &= \lim_{n\to\infty} F(gx_n, gy_n) \\ &= F\left(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n\right) \\ &= F(x, y)(\textbf{??}) \end{aligned} \tag{26}$$

and

$$\begin{aligned} gy &= \lim_{n\to\infty} g(gy_{n+1}) \\ &= \lim_{n\to\infty} F(gy_n, gx_n) \\ &= F\left(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n\right) \\ &= F(y, x)(\textbf{??}) \end{aligned} \tag{27}$$

Hence, we have proved that F and g have a coupled coincidence point.

Now, in the following theorem, we remove the continuity of the map F.

Definition 3.2. Let $(X, \leq)$ be a partially ordered metric space and S be the metric on X. We say that $(X, S, \leq)$ is regular if the following conditions hold:

(i) If a nondecreasing sequence $a_n \to a$ then $a_n \leq a$ for all n.
(ii) If a nondecreasing sequence $b_n \to b$ then $b \leq b_n$ for all n.

Theorem 3.3. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric S on X such that $(X, S, \leq)$ is regular. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} \psi(S(F(x,y), F(x,y), F(u,v))) \leq \frac{1}{2}\psi[(S(gx, gx, gu) + S(gy, gy, gv))] \\ - \phi(S(gx, gx, gu) + S(gy, gy, gv)) \end{aligned}$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$. Suppose that $F(X \times X) \subseteq g(X)$, $g(X)$ is complete. If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$F(x, y) = gx \quad \textit{and} \quad F(y, x) = gy$$

that is, F and g coupled coincidence point.

Proof. Proceeding exactly as in Theorem 3.1, we get $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the complete metric space $g(X)$. Then there exist $x, y \in X$ such that

$$gx_n \to gx \quad \text{and} \quad gy_n \to gy\,. \tag{28}$$

Since $\{gx_n\}$ is nondecreasing and $\{gy_n\}$ is nonincreasing, then since $(X, S, \leq)$ is regular, so we have

$$gx_n \leq gx \quad \text{and} \quad gy_n \geq gy \quad \text{for all } n.$$

If $gx_n = gx$ and $gy_n = gy$ for all $n > 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n$ and $gy \leq gy_{n+1} \leq gy_n = gy$ which gives us that

$$gx_n = gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy = gy_{n+1} = F(y_n, x_n)$$

that is (x_n, y_n) is a coupled coincidence point of F and g. Thus, we suppose $(gx_n, gy_n) \neq (gx, gy)$ for all $n > 0$. Now, using equation (1), consider

$$\begin{aligned}
&\psi(S(gx, gx), F(x, y)) \\
&\leq \psi(2S(gx, gx, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, F(x, y))) \\
&\leq \psi(2S(gx, gx, gx_{n+1}) + \psi(S(F(x_n, y_n), F(x_n, y_n), F(x, y))) \\
&\leq \psi(2S(gx, gx, gx_{n+1})) + \frac{1}{2}\psi(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)) \\
&\qquad + \varphi(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy))
\end{aligned} \tag{29}$$

Letting $n \to \infty$ and using (28), then the right hand side of equation (29) tends to 0, thus $\psi(S(gx, gx, F(x, y)) = 0$. Now, by the property (i) of ψ, we have $(S(gx, gx, F(x, y)) = 0$. It gives that $g(x) = F(x, y)$.

Similarly, $gy = F(y, x)$.

Hence, we have shown that F and g have a coupled coincidence point. □

Corollary 3.4. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric S on X such that (X, S) is a complete S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}
\psi(S(F(x, y), F(x, y), F(u, v))) \leq{}& \frac{1}{2}\psi[\max S(gx, gx, gu), S(gy, gy, gv)] \\
&- \phi(S(gx, gx, gu) + S(gy, gy, gv))
\end{aligned}$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$F(x, y) = gx \quad \textit{and} \quad F(y, x) = gy$$

that is, F and g coupled coincidence point.

Proof. Since, we know that

$$\max S(gx,gx,gu), S(gy,gy,gv) \leq S(gx,gx,gu) + S(gy,gy,gv)$$

then we can apply Theorem 3.1, since ψ is assumed to be nondecreasing. □

Similarly, as an easy consequence of Theorem 3.3, we obtain the following corollary.

Corollary 3.5. *Let $(X, \leq)$ be a partially ordered set and assume that there is a metric S on X such that $(X, S, \leq)$ is regular. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Suppose also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi[S(F(x,y), F(x,y), F(u,v))] \leq \frac{1}{2}\psi\{\max\{S(gx,gx,gu), S(gy,gy,gv))\} - \phi(S(gx,gx,gu) + S(gy,gy,gv))$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$.

Suppose that $F(X \times X) \subseteq g(X)$, $g(X)$ is complete metric space. If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$F(x,y) = gx \quad \textit{and} \quad F(y,x) = gy$$

that is, F and g have a coupled coincidence point.

Proof. Since,

$$\max S(gx,gx,gu), S(gy,gy,gv) \leq S(gx,gx,gu) + S(gy,gy,gv)\,.$$

So, we can apply Theorem 3.1, since ψ is assumed to be nondecreasing. □

Corollary 3.6. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric S on X such that (X, S) is a S-metric space. Assume that $F : X \times X \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Suppose also that there exists $k \in [0, 1)$ such that*

$$S(F(x,y), F(x,y), F(u,v)) \leq \frac{k}{2}[S(gx,gx,gu) + S(gy,gy,gv)]$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$F(x,y) = gx \quad \textit{and} \quad F(y,x) = gy$$

that is, F and g have a coupled coincidence point.

Proof. It is sufficient to set $\psi(t) = t$ and $\phi(t) = \frac{1-k}{2}t$ in Theorem 3.1. □

Corollary 3.7. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric S on X such that $(X, S, \leq)$ is regular. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Suppose also that there exists $k \in [0, 1)$ such that*

$$S(F(x, y), F(x, y), F(u, v)) \leq \frac{k}{2} S(gx, gx, gu) + S(gy, gy, gv)$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$.

Suppose that $F(X \times X) \subseteq g(X)$, $g(X)$ is complete metric space. If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$F(x, y) = gx \quad \textit{and} \quad F(y, x) = gy$$

that is, F and g have a coupled coincidence point.

Proof. It is sufficient to take $\psi(t) = t$ and $\phi(t) = \frac{1-k}{2} t$ in Theorem 3.3. □

Corollary 3.8. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric S on X such that (X, S) is complete S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exists $k \in [0, 1)$ such that*

$$S(F(x, y), F(x, y), F(u, v)) \leq \frac{k}{2} S(gx, gx, gv) + S(gy, gy, gv)$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$F(x, y) = gx \quad \textit{and} \quad F(y, x) = gy.$$

Proof. We know that

$$\max S(gx, gx, gu), S(gy, gy, gv) \leq S(gx, gx, gu) + S(gy, gy, gv) .$$

Then we can apply here corollary 3.8 and obtain the proof. □

Corollary 3.9. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric S on X such that $(X, S, \leq)$ is regular. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Assume also that there exists $k \in [0, 1)$ such that*

$$S(F(x, y), F(x, y), F(u, v)) \leq \frac{k}{2} \max[S(gx, gx, gv), S(gy, gy, gv)]$$

for any $x, y, u, v \in X$, for which $gx \leq gu$ and $gy \leq gv$.

Suppose that $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete metric space. If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ *such that*

$$F(x,y) = gx \quad and \quad F(y,x) = gy.$$

That is, F *and* g *has a coupled coincidence point.*

Now, we shall show the existence and uniqueness of a coupled common fixed point.

For a product $X \times X$ of a partial ordered set $(X, \leq)$ we define a partial ordering as following:

For all $(x,y),(u,v) \leftarrow X \times X$.

$$(x,y) \leq (u,v) \Rightarrow x \leq u,\ y \geq v(\mathbf{??}) \tag{30}$$

We can say that (x,y) and (u,v) are comparable it $(x,y) \leq (u,v)$ or $(u,v) \leq (x,y)$.

Also, we say that (x,y) is equal to (u,v) if and only if $x = u$ and $y = v$.

Theorem 3.10. *In addition to the hypotheses of Theorem 3.1, suppose that for all* $(x,y),(u,v) \in X \times X$ *there exist* $(a,b) \in X \times X$ *such that*

$$(F(a,b), F(b,a)) \text{ is comparable to } (F(x,y), F(y,x)) \text{ and } (F(u,v), F(v,u)).$$

Then F *and* g *have a unique coupled common fixed point* (x,y) *such that* $x = gx = F(x,y)$ *and* $y = gy = F(y,x)$.

Proof. By Theorem 3.1, we know that the set of coupled coincidence points of F and g is not void. Now, assume that (x,y) and (u,v) are two coupled coincidence points of F and g i.e.

$$F(x,y) = gx, \quad F(u,v) = gu$$

and

$$F(y,x) = gy, \quad F(v,u) = gv\,.$$

Now, we shall prove that (gx, gy) and (gu, gv) are equal. By supposition, there exist $(a,b) \in X \times X$ such that $(F(a,b), F(b,a))$ is comparable to $(F(x,y), F(y,x))$ and $(F(u,v), F(v,u))$. Define sequence $\{ga_n\}$ and $\{gb_n\}$ such that $a_0 = a$, $b_0 = b$ and for any $n \geq 1$

$$ga_n = F(a_{n-1}, b_{n-1}) \text{ and } gb_n = F(b_{n-1}, a_{n-1}) \quad \text{for all } n. \tag{31}$$

Further, set $x_0 = x$, $y_0 = y$ and $u_0 = u$, $v_0 = v$ and in the same fashion define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gu_n\}$ and $\{gv_n\}$. Then

$$\begin{aligned} &gx_n = F(x_{n-1}, y_{n-1}), \quad gu_n = F(u_{n-1}, v_{n-1}) \text{ and} \\ &gy_n = F(y_{n-1}, x_{n-1}), \quad gv_n = F(v_{n-1}, u_{n-1}) \text{ for all } n \geq 1. \end{aligned} \tag{32}$$

Since $(F(x,y), F(y,x)) = (gx_1, gu_1) = (gx, gy)$ is comparable to $(F(a,b), F(b,a)) = (ga_1, gb_1)$.

The, it is easy to show $(gx, gy) \geq (ga_1, gb_1)$.

By continuing this, we have

$$(ga_n, gb_n) \leq (gx, gy), \quad \text{for all } n. \tag{33}$$

$$\begin{aligned}\psi(S(ga_{n+1}, ga_{n+1}, gx)) &= \psi(S(F(a_n, b_n), F(a_n, b_n), F(x, y)) \\ &\le \frac{1}{2}\psi[S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy)] \\ &\quad - \phi(S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy)) \qquad (34)\end{aligned}$$

and

$$\begin{aligned}\psi(S(gy, gy, gb_{n+1})) &= \psi(S(gb_{n+1}, gb_{n+1}, gy)) \\ &= \psi(S(F(b_n, a_n), F(b_n, a_n), F(y, x)) \\ &\le \frac{1}{2}\psi[S(gb_n, gb_n, gy) + S(ga_n, ga_n, gx)] \\ &\quad - \phi(S(gb_n, gb_n, gy) + S(ga_n, ga_n, gx)) \qquad (35)\end{aligned}$$

From equation (35) and (36), we have

$$\begin{aligned}&\psi(S(ga_{n+1}, ga_{n+1}, gx)) + \psi(S(gb_{n+1}, gb_{n+1}, gy)) \\ &\le \psi[S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy)] \\ &\quad - 2\phi(S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy)) \qquad (36)\end{aligned}$$

Now, from the property of (iii) of ψ, we get that

$$\begin{aligned}&\psi(S(ga_{n+1}, ga_{n+1}, gx) + S(gb_{n+1}, gb_{n+1}, gy)) \\ &\le \psi[S(ga_{n+1}, ga_{n+1}, gx) + S(gb_{n+1}, gb_{n+1}, gy)] \\ &\le \psi[S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy)] \\ &\quad - 2\phi(S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy))\,. \qquad (37)\end{aligned}$$

Now, let $\sigma_n = S(ga_n, ga_n, gx) + S(gb_n, gb_n, gy)$.

Then from equation (37), we get

$$\psi(\sigma_{n+1}) = \psi(\sigma_n) - 2\phi(\sigma_n) \quad \text{for all } n \qquad (38)$$

which gives us that $\psi(\sigma_{n+1}) \le \psi(\sigma_n)$. By the property of ψ, we get that $\sigma_{n+1} \le \sigma_n$. So the sequence $\{\sigma_n\}$ is decreasing and bounded below from 0. So, there exists $\sigma \ge 0$ such that

$$\lim_{n\to\infty} \sigma_n = 0.$$

Now, we shall show that $\sigma = 0$. Assume on the contrary that $\sigma > 0$. Taking $n \to \infty$ in equation (38), we get that

$$\psi(\sigma) \le \psi(\sigma) - u \lim_{n\to\infty} \psi(\sigma_n) < \psi(\sigma)$$

which gives a contradiction. It gives that $\sigma = 0$ that is, $\lim_{n\to\infty} \sigma_n = 0$.

Consequently, we have

$$\lim_{n\to\infty} S(ga_n, ga_n, gx) = 0 \quad \text{and} \quad \lim_{n\to\infty} S(gb_n, gb_n, gy) = 0 \qquad (39)$$

In the same way, we can show that

$$\lim_{n\to\infty} S(ga_n, ga_n, gu) = 0 \quad \text{and} \quad \lim_{n\to\infty} S(gb_n, gb_n, gv) = 0 \qquad (40)$$

Combining equations (39) and (40), gives that (gx, gy) and (gu, gv) are equal.

Since $gx = F(x, y)$, $gy = F(y, x)$ and by the commutativity of F and g, we get that

$$gx' = g(gx) = g(F(x, b)) = F(gx, gy) = F(x', y')$$
$$gy' = g(gy) = g(F(y, x)) = F(gy, gx) = F(y', x')$$

where $gx = x'$, $gy = y'$. Thus (x', y') is a coupled coincidence point of F and g. So (gx', gy') and (gx, gy) are equal, we get that

$$gx' = gx = x', \quad gy' = gy = y' .$$

Thus, (x', y') is a coupled common fixed point of F and g. Its uniqueness follows from contradiction in this theorem. □

Example 3.11. Let $X = \mathbb{R}$ with the metric

$$S(x, y, z) = |x - y - z| \text{ for all } x, y, z \in X \text{ and the usual ordering.}$$

Let $F : X \times X \to X$ and $g : X \to X$ be defined as

$$gx = \frac{x}{2}, \quad F(x, y) = \frac{x - y}{8} \text{ for all } x, y \in X.$$

Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \frac{t}{10}, \quad \phi(t) = \frac{t}{30} \text{ for all } t \in [0, \infty) .$$

It is easy to check that all condition of Theorem 3.10 are satisfied for all $x, y, u, v \in X$ satisfying $gx \leq gv$ and $gv \geq gy$. This we have, $(0, 0)$ is the unique coupled fixed point of F and g.

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