Fourier Transform and Invariant differential Operators on Galilean Group

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Abstract

Let G be the $10-$ dimensional Galilean group, and Let $\mathcal{U}$ be the algebra of all the invariant partial differential equations on G. As well known Lorentz invariance and Galilean invariance are nearly the same. Maxwell,s equations are invariant by Lorentz group and the principle of Galilean invariance says that the laws of physics are invariant under the Galilean transformations. So it is interesting to study the algebra $\mathcal{U}$. For this, we will use the Fourier transform on G in order to solve the elements of invariant differential equations on G.

Keywords: Galilean Group, Semidirect Product of Two Lie Groups, Fourier Transform, Invariant Differential Operators.

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1 Notations and Results.

1. The Galilean group G is non commutative Lie group of the $10-$ parameters of transformations in the four dimensional (t, x, y, z), and is one of the most important group in quantum physics. The mathematical essence of the special relativity is the Galilean group is replaced by the Lorentz group. The principle of Galilean invariance says that the laws of physics are invariant under the Galilean transformations.The concept of kinetic energy is associated with motion, it is the direct consequence of the Galilean invariance applied to a material point. For $v \ll c$ the relatively small speed the Galilean transformations still hold (Einstein 1916), where c is the speed of light. Besides there are other systems of equations which include the first pair of Maxwell,s equations and are invariant under the Glilean group. So one of the intersting branch in mathematical physics is the Fourier transform on the non commutative Lie groups and which is the most widely used mathematical tools in engineering for solving problems in robotics, image analysis, computer vision , mechanics. For a long time, people have tried to construct objects in order to generalize Fourier transform and Pontryagin,s theorem to the non abelian case. However, with the dual object not being a group, it is not possible to define the Fourier transform and the inverse of the Fourier transform between G and $\widehat{G}$. The goal of this paper is to define the Fourier transform on G, by combining the classical Foureir transform on $\mathbb{R}^n$ and on a connected compact Lie group K, and then we prove the necessary and sufficient condition for the solvability of an invariant differential operators on G.

Definition 1.1. *The special Galilean group is a $10-$dimensional Lie group and defined by*

$$X = \begin{bmatrix} r & x & y \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^3, y \in \mathbb{R}^3$ and $r \in SO(3)$. The multiplication of two elements X and Y is given by

$$\begin{aligned} & X.Y \\ = & \begin{bmatrix} r_1 & x_1 & y_1 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_2 & x_2 & y_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} r_1 r_2 & r_1 x_2 + x_1 & r_1 y_2 + x_1 t_2 + y_1 \\ 0 & 1 & t_1 + t_2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{2}$$

and the inverse of any element X

$$X^{-1} = \begin{bmatrix} r_1^{-1} & -r_1^{-1} x_1 & -r_1^{-1} y_1 + r_1^{-1}(x_1 t_1) \\ 0 & 1 & -t_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

1.2. If we refer to [8, 9, 10, 11], then it is easy to show that the group $G \simeq H \rtimes_{\rho_1} SO(3)$ is the semidirect product of the two groups H and $SO(3)$, where $SO(3)$ is the group of rotation of $\mathbb{R}^3$,and H is the group consisting of all matrix of the form

$$H = \{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, x \in \mathbb{R}^3, y \in \mathbb{R}^3, t \in \mathbb{R} \} \tag{4}$$

where

$$\rho_1(r)((y,t),x) = ((ry,t),rx) \tag{5}$$

for all $(y,t) \in \mathbb{R}^4$, $x \in \mathbb{R}^3$ and $r \in SO(3)$. Also the group $H = \mathbb{R}^4 \rtimes_{\rho_2} \mathbb{R}^3$ is the semidirect of the two vector groups $\mathbb{R}^4$ and $\mathbb{R}^3$, where

$$\rho_2(x)(y,s) = (y + xs, x) \tag{6}$$

for all $(y,t) \in \mathbb{R}^4$, and $x \in \mathbb{R}^3$. So the myltiplication of two elements X and Y belong to G can be shown as

$$
\begin{aligned}
& X.Y = \\
= \quad & ((y_1, t_1), x_1, r_1)(y_2, t_2, x_2, r_2) \\
= \quad & ((y_1, t_1), x_1)(\rho_1(r_1)(y_2, t_2, x_2)), r_1 r_2) \\
= \quad & ((y_1, t_1), x_1)((r_1 y_2, t_2), r_1 x_2), r_1 r_2) \\
= \quad & (((y_1, t_1) + \rho_2(x_1)(r_1 y_2, t_2)), x_1 + r_1 x_2, r_1 r_2) \\
= \quad & (((y_1, t_1) + (r_1 y_2 + t_2 x_1, t_2)), x_1 + r_1 x_2, r_1 r_2) \\
= \quad & ((y_1 + r_1 y_2 + t_2 x_1, t_1 + t_2), x_1 + r_1 x_2, r_1 r_2) \qquad (7)
\end{aligned}
$$

1.3. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group G, which are integrable with respect to the Haar measure of G and multiplication is defined by convolution on G , and we denote by $L^2(G)$ the Hilbert space of G. So we have for any $f \in L^1(G)$ and $\phi \in L^1(G)$

$$\phi * f(X) = \phi * f(X) = \int\limits_G f(Y^{-1}X)\phi(Y)dY \qquad (8)$$

where $dY = dy_7 dy_6 dy_5 dy_4 dy_3 dy_2 dy_1 dr$ is the Haar measure on G which is the Haar measure on G, $Y = (y_7, y_6, y_5, y_4, y_3, y_2, y_1, r)$, $X = (x_7, x_6, x_5, x_4, x_3, x_2, x_1, q)$ and $*$ denotes the convolution product on G. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra of G; which is canonically isomorphic onto the algebra of all distributions on G supported by $\{I_G\}$, where $\{I_G\}$ is the identity element of G. For any $u \in \mathcal{U}$ one can define an left invariant differential operator P on G as follows:

$$P_u f(X) = u * f(X) = \int\limits_G f(Y^{-1}X)\phi(Y)dY \qquad (9)$$

The mapping $u \to P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on G. For more details see [4, 13]. Let $A = \mathbb{R}^4 \times \mathbb{R}^3 \times SO(3)$ be the group of the direct product of $\mathbb{R}^4$, $\mathbb{R}^3$ and $SO(3)$, then we get for any $f \in L^1(G)$ and $\phi \in L^1(G)$ as follows

$$\phi * f(X) = \phi *_c f(V) = f *_c \phi(V) = \int\limits_A f(V - W, rs^{-1})\phi(W)dWds \qquad (10)$$

where $*_c$ means the convolution product on the group A, $V = (v_7, v_6, v_5, v_4, v_3, v_2, v_1) \in \mathbb{R}^7$, $W = (w_7, w_6, w_5, w_4, w_3, w_2, w_1) \in \mathbb{R}^7$, $r \in SO(3)$, $s \in SO(3)$, and $dW = dw_7 dw_6 dw_5 dw_4 dw_3 dw_2 dw_1$ is the Haare measure on A. We denote by $\mathcal{U}_A$ the complexified universal enveloping algebra of the real Lie algebra of A,which is canonically isomorphic onto the algebra of all distributions on A supported by the $\{I_A\}$, where $\{I_A\}$ is the identity element of A. For any $u \in \mathcal{U}$ one can define a differential operator with constant coefficients P on A as

$$Qf(V) = f *_c Q_u(V) = \int_A f(V - W, rs^{-1})\phi(W)dWds \tag{11}$$

2 Fourier Transform on the Galilean Group G.

2. Consider the set $K = \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times SO(3)$ with law:

$$\begin{aligned}
& X.Y = ((y_1, t_1), z_1, x_1, s_1, r_1)((y_2, t_2), z_2, x_2, s_2, r_2) \\
= \;& (((y_1, t_1), z_1, x_1, s_1)(\rho_1(r_1)((y_2, t_2), z_2, x_2, s_2)), r_1 r_2) \\
= \;& ((((y_1, t_1), z_1, x_1, s_1)((r_1 y_2, t_2), z_2, r_1 x_2, s_2)), r_1 r_2) \\
= \;& (((((y_1, t_1), z_1, x_1)(\rho_2(x_1)((r_1 y_2, t_2), z_2, r_1 x_2))), s_1 s_2), r_1 r_2) \\
= \;& ((y_1, t_1) + (r_1 y_2 + t_2 x_1, t_2), z_1 + z_2, x_1 + r_1 x_2, s_1 s_2, r_1 r_2) \\
= \;& (y_1 + r_1 y_2 + t_2 x_1, t_1 + t_2, z_1 + z_2, x_1 + r_1 x_2, s_1 s_2, r_1 r_2) \qquad (12)
\end{aligned}$$

for all $X \in K$ and $Y \in K$. In this case the group G can be identified with the closed subgroup $\mathbb{R}^4 \times \{0\} \rtimes_{\rho_2} \mathbb{R}^3 \times \{I\} \rtimes_{\rho_1} SO(3)$ of K and A with the closed subgroup $\mathbb{R}^4 \times \mathbb{R}^3 \times \{0\} \times SO(3) \times \{I\}$ of K, where $\{0 = (0,0,0)\}$ is the identity elemet of $\mathbb{R}^3$ and $\{I\}$ is the idetity element of $SO(3)$.

Definition 2.1. *For every $f \in L^1(G)$, one can define function $\widetilde{f}$* on K *as*

$$\widetilde{f}((y,t), z, x, s, r) = f((\rho_1(s)((\rho_2(z)(y,t)), z + x), rs)) \tag{13}$$

for all $((y,t), z, x, s, r) \in K = \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times SO(3)$.

Remark 2.1. *The function $\widetilde{f}$ is invaraiant in the following sens*

$$\begin{aligned}
& \widetilde{f}((\rho_1(h)((\rho_2(k)((y,t), u - k, v + k)))), qh^{-1}, hr) \\
= \;& \widetilde{f}((y,t), u, v, q, r) \qquad (14)
\end{aligned}$$

So every function $\varphi((y,t),x,r)$ on G extends uniquely as an invariant function $\varphi((y,t),z,x,s,r)$ on K.

Theorem 2.1. *For every function $\varphi \in L^1(K)$ invariant in sense (14) and for every $\phi \in L^1(G)$, we have*

$$\phi * \varphi((y,t),u,v,q,r) = \phi *_c \varphi((y,t),u,v,q,r) \tag{15}$$

for every $((y,t),u,v,q,r) \in K$, where $*$ signifies the convolution product on G with respect the variables $((y,t),v,r)$ and $*_c$signifies the commutative convolution product on A with respect the variables $((y,t),u,q)$.

Proof: Since φ is invariant in sense (14), then for every $((x,s),u,v,q,r) \in K$, we get

$$\begin{aligned}
& \phi * \varphi((y,t),u,v,q,r) \\
= & \int\limits_{GA} \varphi\left[((y',t'),v',r')^{-1}((y,t),u,v,q,r)\right]\phi(x',s',v',r')dy'dt'd\acute{v}dr' \\
= & \int\limits_{GA} \varphi[((\rho_1(r'^{-1})((y',t'),v')^{-1})),r'^{-1})((y,t),u,v,q,r)]\phi(x',s',v',r')dy'dt'd\acute{v}dr' \\
= & \int\limits_{GA} \varphi[((\rho_1(r'^{-1})((y',t'),v')^{-1}((y,t),u,v)),q,r'^{-1}r)]\phi(x',s',v',r')dy'dt'd\acute{v}dr' \\
= & \int\limits_{GA} \varphi[((\rho_1(r'^{-1})(\rho_2(-v')(y',t')^{-1}(y,t),u,v-v')),q,r'^{-1}r]\phi(x',s',v',r')dy'dt'd\acute{v}dr' \\
= & \int\limits_{GA} \varphi[((\rho_1(r'^{-1})(y-y',t-t'),u-v'\ ,v)),q,r'^{-1}r]\phi(x',s',v',r')dy'dt'd\acute{v}dr' \\
= & \int\limits_{GA} \varphi[(y-y',t-t'),u-v'\ ,v,qr'^{-1},r]\phi(y',t',v',r')dy'dt'd\acute{v}dr' \\
= & \ \phi *_c \varphi
\end{aligned} \tag{16}$$

Let $\widehat{SO(3)}$ be the set of all irreducible unitary representations of $SO(3)$. If $\gamma \in \widehat{SO(3)}$, we denote by E_γ the space of representaion of γ and d_γ its demension

Definition 2.2. *The Fourier transform of a function $f \in C^\infty(SO(3))$ is* *defined as*

$$\widehat{f}(\gamma) = \int\limits_{SO(3)} f(x)\gamma(x^{-1})dx \tag{17}$$

Theorem 2.2. *Let $f \in L^1(SO(3)) \cap L^2(SO(3))$, then we have the inversion of the Fourier transform*

$$f(x) = \sum_{\gamma \in \widehat{SO(3)}} d_\gamma tr[\widehat{f}(\gamma)\gamma(x)] \tag{18}$$

$$f(I) = \sum_{\gamma \in \widehat{SO(3)}} d_\gamma tr[\widehat{f}(\gamma)] \tag{19}$$

and the Plancherel formula

$$\|f(x)\|^2 = \sum_{\gamma \in \widehat{SO(3)}} d_\gamma \left\|\widehat{f}(\gamma)\right\|_2^2 \tag{20}$$

where $\left\|\widehat{f}(\gamma)\right\|_2^2$ is the norm of Hilbert-Schmidt of the operator $\widehat{f}(\gamma)$, see [2]

Definition 2.2. *If $f \in L^1(G)$, we define the Fourier transform of f as follows*

$$T\mathcal{F}\ f((\xi,\lambda),\mu,\gamma) = \int\limits_{\mathbb{R}^7}\int\limits_{SO(3)} f((y,t),x,r)e^{-\,i\,\langle((\xi,\lambda),\eta),(y,s,x)\rangle}\gamma(r^{-1})dydtdxdr \tag{21}$$

where $(y,t) = (y_1,y_2,y_3,t) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$, $x = (x_1,x_2,x_3)$, $\xi = (\xi_1,\xi_2,\xi_3,\lambda) \in \mathbb{R}^4$, $\lambda \in \mathbb{R}$, $\eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3$, $dx = dx_1dx_2dx_3$, $dy = dy_1dy_2dy_3$, $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^4 \times \mathbb{R}^3$ and T is the Fourier transform on $SO(3)$.

Definition 2.3. *If $f \in L^1(G)$, we define the Fourier transform of its* *invariant* $\widetilde{f}$ *as follows*

$$\begin{aligned}
& T\mathcal{F}\ \widetilde{f}(\xi,\lambda,\eta,0,\gamma,I) \\
= & \int\limits_{\mathbb{R}^7}\int\limits_{\mathbb{R}^3}\int\limits_{SO(3)}\sum_{\delta\in\widehat{SO(3)}} d_\delta tr[\int\limits_{SO(3)}\widetilde{f}((y,t),u,v,q,r)\delta(r^{-1})dr]\gamma(q^{-1})dq \\
& e^{-\,i\,\langle((\xi,\lambda),\eta),((x,s),u)\rangle}e^{-\,i\,\langle\mu,v\rangle}dydtdvd\mu \\
= & \int\limits_{\mathbb{R}^7}\int\limits_{\mathbb{R}^3}\int\limits_{SO(3)}\sum_{\delta\in\widehat{SO(3)}} d_\delta tr[\widetilde{f}((x,s),u,v,q,\delta)]\gamma(q^{-1})dq \\
& e^{-\,i\,\langle((\xi,\lambda),\eta),((x,s),u)\rangle}e^{-\,i\,\langle\mu,v\rangle}dydtdvd\mu \qquad (22)
\end{aligned}$$

Proposition 2.1. *For every* $\phi\in L^1(G)$, *and* $f\in L^1(G)$, *we have*

$$\begin{aligned}
& T\mathcal{F}\ (\phi*\widetilde{f})(\xi,\lambda,\eta,0,\gamma,I) = T\mathcal{F}\ (\phi*_c\widetilde{f})(\xi,\lambda,\eta,0,\gamma,I) \\
= & \ T\mathcal{F}(\widetilde{f})(\xi,\lambda,\eta,0,\gamma,I)\mathcal{F}(\phi)(\xi,\lambda,\eta,\gamma) \qquad (23)
\end{aligned}$$

Proof: By (15) we have

$$\begin{aligned}
& \int\limits_{\mathbb{R}^3} T\mathcal{F}(\phi*\widetilde{f})(\xi,\lambda,\eta,\mu,\gamma,I)d\mu \\
= & \int\limits_{\mathbb{R}^3} T\mathcal{F}(\phi*_c\widetilde{f})(\xi,\lambda,\eta,\mu,\gamma,I)d\mu \\
= & \int\limits_{\mathbb{R}^3} T\mathcal{F}(\phi)(\xi,\lambda,\eta,\gamma)T\mathcal{F}(\widetilde{f})(\xi,\lambda,\eta,\mu,\gamma,I)d\mu \\
= & \ T\mathcal{F}(\widetilde{f})(\xi,\lambda,\eta,0,\gamma,I)T\mathcal{F}(\phi)(\xi,\lambda,\eta,\gamma)
\end{aligned}$$

Theorem 2.3. **(*Plancherel formula*).** *For any* $f\in\ L^1(G)\cap\ L^2(G)$, *we get*

$$\begin{aligned}
& f\ *\widetilde{f}(0,0,0,0,I,I) = \int\limits_{\mathbb{R}^7}\sum_{\gamma\in\widehat{SO(3)}} d_\gamma\left\|f((y,t),v,\gamma)\right\|_2^2 dydtdv \\
= & \int\limits_{\mathbb{R}^7}\sum_{\gamma\in\widehat{SO(3)}} d_\gamma\left\|T\mathcal{F}f(\xi,\lambda,\eta,\gamma\right\|_2^2 d\xi d\lambda d\eta \qquad (24)
\end{aligned}$$

For the proof of this theorem see [10].

3 Invariant differential operators on G.

3. In the following we will give a necessary and sufficient condition for the elements of

Definition.3.1. *Let P be an invariant differential operator on a connected Lie group H. by dfinitione P is said to be semi-globally solvable if there exist a distribution T on H such that*

$$PT = \delta_H \tag{25}$$

where δ_H is the Dirac measure at the identity element of H.

Definition.3.2. Let *P be an invariant differential operator on a connected Lie group H by definition P is said to be globally solvable if for any function $g \in C^{\infty}(H)$,there exist a function $f \in C^{\infty}(H)$ such that*

$$Pf = g \tag{26}$$

For all the following notations and results, see [2]

Let $\underline{k}$ be the the Lie algebra of the subgroup $SO(3)$ of G and let (Z_1, Z_2, Z_3) be a basis of $SO\,(3)$,such that the following invariant differential operator

$$D_q = \sum_{0 \le l \le q} \left(-\sum_{i=1}^{3} Z_i^2 \right)^l \tag{27}$$

is bi-invariant on $SO(3)$. Let $\widehat{SO(3)}$ be the set of all irreducible unitary representations of $SO(3)$. If $\gamma \in \widehat{SO(3)}$, we denote by E_γ the space of representaion γ and d_γ its demension then we get

$$1 \le \langle D_q tr\gamma(r), tr(r) \rangle = d_q(\gamma) \tag{28}$$

If $u(\eta)$ is a polynomial on $\mathbb{R}^3$ valued in E_γ, we denote by $\widetilde{u}(\eta)$ the matrix defined by

$$[\widetilde{u}_{ij}(\eta)] \tag{29}$$

where

$$\widetilde{u}_{ij}(\eta) = \left(\sum_{|\alpha| \le m} \left| u_{ij}(\eta)^{(\alpha)} \right|^2 \right)^{\frac{1}{2}} \tag{30}$$

$\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $1 \le \alpha_j \le 3$, $\eta^\alpha = \eta_{\alpha_1}......\eta_{\alpha_m}$, and (η_1, η_2, η_3) is basis of $\mathbb{R}^3$. In this case we get.

$$\|\widetilde{u}(\eta)\|^2 = \sum_{|\alpha| \le m} \left\|u(\eta)^{(\alpha)}\right\|^2 \tag{31}$$

Theorem 3.1. *Let P be a right invariant differential on the Galilean group G, then P has a fundamental solution if and only if there is a constant C and a number $q \in N$, such that*

$$\det \ u(\gamma, 0, 0) \neq 0, \ \textit{and} \ \left\| \frac{{}^{co}u((0,0),0,\widetilde{\gamma})}{\det \ u((0,0),0,\widetilde{\gamma})} \right\| \le C d_q(\gamma) \tag{32}$$

for any $\gamma \in \widehat{SO(3)}$

Proof: Let u be the distribution associted to the right invariant differential operator P. Let $End(E_\gamma)$ be the space of all enomorphisms of E_γ and let P_ζ be the polynomial valued in $End(E_\gamma)$, defined by

$$\begin{aligned}(\zeta, \omega, \pi, \gamma) &\mapsto \mathcal{F}(\overset{\vee}{u})((\xi, \lambda), \mu, \gamma) + (\zeta, \omega, \pi, \gamma)) \\ &= \mathcal{F}(\overset{\vee}{u})((\xi + \zeta, \lambda + \omega), \mu + \pi, \gamma)\end{aligned} \tag{33}$$

For any $f \in \mathcal{D}(K)$, we put

$$\begin{aligned}&\langle S, f\rangle \\ = &\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\Omega} \sum_{\gamma \in \widehat{SO(3)}} d_\gamma tr[\frac{{}^{co}\mathcal{F}(\overset{\vee}{u})((\xi+\zeta, \lambda+\omega), \mu+\pi, \gamma)}{\det \ \mathcal{F}(\overset{\vee}{u})((\xi+\zeta, \lambda+\omega), \mu+\pi, \gamma)} \mathcal{F}(f)((\xi+\zeta, \lambda+\omega), \mu+\pi, \gamma)] \\ &\Phi(P_\zeta, \xi) d\zeta d\pi d\omega d\xi d\lambda d\eta\end{aligned} \tag{34}$$

where Ω is a ball in $\mathbb{C}^7$ with center 0, Φ is Hormander,s function[14]. Then by [2, $573 - 579$], S defines a distribution on G.Now we can define a new distribution $\widetilde{S}$ assiciated to S defined by

$$
\begin{aligned}
& \langle \widetilde{S},\ f\ \rangle \\
= \ & \langle S,\ \widetilde{f}\ \rangle \\
= \ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\Omega} \sum_{\gamma\in\widehat{SO(3)}} d_\gamma tr[\frac{{}^{co}\mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)}{\det\ \mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)} \\
& T\mathcal{F}(\widetilde{f})((\xi+\zeta,\lambda+\omega),\mu+\pi,0,\gamma,I))]\Phi(P_\zeta,\xi)d\zeta d\pi d\omega d\xi d\lambda d\eta
\end{aligned}
$$

then we have by $[2,\ 573-579\ ]$ and equation (23) , we have

$$
\begin{aligned}
& \langle \widetilde{u*S},\ f\ \rangle \\
= \ & \langle u*S,\ \widetilde{f}\ \rangle = \langle S, u*\ \widetilde{f}\ \rangle \\
= \ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\Omega} \sum_{\gamma\in\widehat{SO(3)}} d_\gamma tr[\frac{{}^{co}\mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)}{\det\ \mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)} \\
& T\mathcal{F}(\overset{\vee}{u}*\widetilde{f})((\xi+\zeta,\lambda+\omega),\mu+\pi,0,\gamma,I))]\Phi(P_{\zeta,\omega,\pi},\xi,\lambda,\mu)d\zeta d\pi d\omega d\xi d\lambda d\eta \\
= \ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\Omega} \sum_{\gamma\in\widehat{SO(3)}} d_\gamma tr[\frac{{}^{co}\mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)}{\det\ \mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)} \\
& T\mathcal{F}(\overset{\vee}{u}*_c\widetilde{f})((\xi+\zeta,\lambda+\omega),\mu+\pi,0,\gamma,I))]\Phi(P_{\zeta,\omega,\pi},\xi,\lambda,\mu)d\zeta d\pi d\omega d\xi d\lambda d\eta \\
= \ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\Omega} \sum_{\gamma\in\widehat{SO(3)}} d_\gamma tr[\frac{{}^{co}\mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)\mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)}{\det\ \mathcal{F}(\overset{\vee}{u})((\xi+\zeta,\lambda+\omega),\mu+\pi,\gamma)} \\
& T\mathcal{F}(\widetilde{f})((\xi+\zeta,\lambda+\omega),\mu+\pi,0,\gamma,I))]\Phi(P_{\zeta,\omega,\pi},\xi,\lambda,\mu)d\zeta d\pi d\omega d\xi d\lambda d\eta \\
= \ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\Omega} \sum_{\gamma\in\widehat{SO(3)}} d_\gamma tr[T\mathcal{F}(\widetilde{f})((\xi+\zeta,\lambda+\omega),\mu+\pi,0,\gamma,I))] \\
& \Phi(P_{\zeta,\omega,\pi},\xi,\lambda,\mu)d\zeta d\pi d\omega d\xi d\lambda d\eta \\
= \ & \widetilde{f}(0,0,0,0,I,I) = \langle \delta_G, \widetilde{f}\rangle = \langle \widetilde{\delta_G}, f\ \rangle
\end{aligned}
$$

By Hormander construction, and proposition 2.1, we obtain for any $u\in\mathcal{U}$

$$\widetilde{u*S}((y,t),u,0,q,I) = \widetilde{\delta}_G((y,t),u,0,q,I). \tag{35}$$

where δ_L is the Dirac measure at the identity element of L, Consequently, we have

$$u*S((y,t),v,r) = \delta_G((y,t),v,r) \tag{36}$$

Hence the proof of our theorem

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