UPPER AND LOWER WEAKLY LAMBDA CONTINUOUS MULTIFUNCTIONS

R.VENNILA

Department of Mathematics, SriGuru Institute of Technology,Varathaiyangar Palayam, Coimbatore-641 110, India. r.vennila@rediffmail.com

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ABSTRACT: In this paper, a new notion in topological spaces called weakly λ-continuous multifunction is introduced and studied the characterization of weakly λ-continuous multifunctions.

Keywords: Topological spaces, λ-open sets, λ-closed sets, weakly λ-continuous multifunctions.

1. INTRODUCTION AND PRELIMINARIES

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and good numbers of them have been extended to multifunctions. This implies that both functions and multifunctions are important tools in the whole Mathematical Science. Maki [6] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel, that is, to the intersection of all open super sets of A. Arenas et.al. [1] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Caldas et.al. [3] introduced and studied some new notions by utilizing the notion of λ-open sets defined in [1]. Let A be a subset of a topological space (X,τ). The closure and the interior of a set A is denoted by Cl(A), Int(A) respectively. A subset A of a topological space (X,τ) is said to be λ-closed [1] if A = B∩C, where B is a Λ-set and C is a closed set of X. The complement of λ-closed set is called λ-open [3]. A point $x \in X$ in a topological space (X,τ) is said to be λ-cluster point of A [3] if for every λ-open set U of X containing x, $A \cap U \neq \phi$. The set of all λ-cluster points of A is called the λ-closure of A and is denoted by $Cl_\lambda(A)$ [3]. A point $x \in X$ is said to be the λ-interior point of A if there exists a λ-open set U of X containing x such that $U \subset A$. The set of all λ-interior points of A is said to be the λ-interior of A and is denoted by $Int_\lambda(A)$. The family of all λ-open (resp.λ-closed) sets of X is denoted by λO(X) (resp. λC(X)). The family of all λ-open (resp. λ-closed) sets of a space (X,τ) containing the point $x \in X$ is denoted by λO(X, x) (resp. λC(X, x)). Let B be a subset of Y. For a multifunction $F: X \to Y$, upper and lower inverse of any subset B of Y is defined by $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \phi\}$.

2. WEAKLY λ-CONTINUOUS MULTIFUNCTIONS

Definition 2.1: A multifunction $F: X \to Y$ is said to be:

(i) upper weakly λ-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \lambda O(X, x)$ such that $U \subset F^+(Cl(V))$;

(ii) lower weakly λ-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \lambda O(X, x)$ such that $U \subset F^-(Cl(V))$;

(iii) Weakly λ-continuous if F is both upper weakly λ-continuous and lower weakly λ-continuous.

Theorem 2.2: For a multifunction $F: X \to Y$, the following statements are equivalent:

(i) F is upper weakly λ-continuous;

(ii) $F^+(V) \subset Int_\lambda(F^+(Cl(V)))$ for any open set V of Y;

(iii) $Cl_\lambda(F^-(Int(K))) \subset F^-(K)$ for any closed set K of Y;

(iv) for each $x \in X$ and each open set V containing $F(x)$, there exists a λ-open set U containing x such that $F(U) \subset Cl(V)$.

Proof: (i)⇒(ii): Let V be any open set and $x \in F^+(V)$. By (i), there exists a λ-open set U containing x such that $U \subset F^+(Cl(V))$. Hence, $x \in Int_\lambda(F^+(Cl(V)))$.

(ii)⇒(i): Let V be any open set and $x \in F^+(V)$. By (ii), $x \in F^+(V) \subset F^+(Cl(V)) \subset Int_\lambda(F^+(Cl(V)))$. Take $U = Int_\lambda(F^+(Cl(V)))$. Thus, we obtain that F is upper weakly λ-continuous.

(ii)⇒(iii): Let K be any closed set of Y . Then, $Y - K$ is an open set By (ii), $F^+(Y-K) = X - F^-(K) \subset Int_\lambda(F^+(Cl(Y-K))) = Int_\lambda(F^+(Y - Int(K)))$. Thus, $Cl_\lambda(F^-(Int(K))) \subset F^-(K)$. The converse is similar.

(i)⇒(iv): Obvious.

Definition 2.3: [4] A multifunction $F : X \to Y$ is said to be:

(i) upper λ-continuous at a point $x \in X$ if for each open set V of Y such that $F(x) \subset V$, there exists $U \in \lambda O(X, x)$ such that $F(U) \subset V$;

(ii) lower λ-continuous at a point $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \phi$, there exists $U \in \lambda O(X, x)$ such that $F(u) \cap V \neq \phi$ for every $u \in U$;

(iii) upper (lower) λ-continuous if F has this property at each point of X.

Remark 2.4: It is clear that every upper λ-continuous multifunction is upper weakly λ-continuous. But the converse is not true in general, as the following example shows.

Example 2.5: Let $X = \{a, b, c\}, \tau = \{\phi, \{a, b\}, X\}$. Then the multifunction $F : (X, \tau) \to (X, \tau)$ defined by $F(a) = \{a\}$, $F(b) = \{c\}$, $F(c) = \{b\}$ is upper weakly λ-continuous but not upper λ-continuous.

Theorem 2.6: For a multifunction $F : X \to Y$, the following statements are equivalent:

(i) F is lower weakly λ-continuous;

(ii) $F^-(V) \subset Int_\lambda(F^-(Cl(V)))$ for any open set V of Y;

(iii) $Cl_\lambda(F^+(Int(K))) \subset F^+(K)$ for any closed set K;

(iv) for each $x \in X$ and each open set V such that $F(x) \cap V \neq \phi$, there exists a λ-open set U containing x such that $F(u) \cap Cl(V) \neq \phi$ for each $u \in U$.

Proof: It is similar to the proof of the Theorem 2.2.

Theorem 2.7: Let $F : X \to Y$ be a multifunction such that $F(x)$ is an open set of Y for each $x \in X$. Then F is lower λ-continuous if and only if lower weakly λ-continuous.

Proof: Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \phi$). Then there exists a λ-open set U containing x such that $F(u) \cap Cl(V) \neq \phi$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \phi$ for each $u \in U$ and hence F is lower λ-continuous. The converse follows by Remark 2.4.

Definition 2.8: A topological space (X, τ) is said to be normal if for every disjoint closed sets V_1 and V_2 of X, there exist two disjoint open sets U_1 and U_2 such that $V_1 \subset U_1$ and $V_2 \subset U_2$.

Theorem 2.9: If $F : X \to Y$ is upper weakly λ-continuous and satisfies the condition $F^+(Cl(V)) \subset F^+(V)$ for every open set V of Y, then F is upper λ-continuous.

Proof: Let V be any open set of Y. Since F is weakly λ-continuous, we have $F^+(V) \subset Int_\lambda(F^+(Cl(V)))$ and hence $F^+(V) \subset Int_\lambda(F^+(Cl(V))) \subset Int_\lambda(F^+(V))$. Thus, $F^+(V)$ is λ-open and it follows that F is upper λ-continuous.

Theorem 2.10: Let $F : X \to Y$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then F is upper weakly λ-continuous if and only if F is upper λ-continuous.

Proof: Suppose that F is upper weakly λ-continuous. Let $x \in X$ and G be an open set containing $F(x)$. Since $F(x)$ is closed in Y and Y is normal, there exist open sets V and W such that $F(x) \subset V$, $X - G \subset W$ and $V \cap W = \phi$. We have $F(x) \subset V \subset Cl(V) \subset Cl(X - W) = X - W \subset G$. Since F is upper weakly λ-continuous, there exists a λ-open set U containing x such that $F(U) \subset Cl(V) \subset G$. This shows that F is upper λ-continuous. The converse follows by Remark 2.4.

Lemma 2.11 : [5] If A is an α-paracompact α-regular set of a topological space (X, τ) and U an open neighbourhood of A, then there exists an open set G of X such that $A \subset G \subset Cl(G) \subset U$.

For a multifunction $F: X \to Y$, by $\mathrm{Cl}\,F: X \to Y$ we denote a multifunction as follows: $\mathrm{Cl}\,F(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we denote $\mathrm{Cl}_\lambda F$.

Lemma 2.12: If $F: X \to Y$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then we have the following:

(i) $G^+(V) = F^+(V)$ for each open set V of Y,

(ii) $G^-(V) = F^-(V)$ for each closed set V of Y, where G denotes ClF or $\mathrm{Cl}_\lambda F$.

Proof: (i): Let V be any set of Y and $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Then we have $F(x) \subset V$ and by Lemma 2.11 there exists an open set V such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since $F^+(V) \subset G^+(V)$, $G^+(V) = F^+(V)$.

(ii): Follows from (i).

Lemma 2.13: For a multifunction $F: X \to Y$, we have the following:

(i) $G^-(V) = F^-(V)$ for each open set V of Y,

(ii) $G^+(V) = F^+(V)$ for each open set V of Y, where G denotes ClF or $\mathrm{Cl}_\lambda F$.

Theorem 2.14: Let $F: X \to Y$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for every $x \in X$. Then the following properties are equivalent:

(i) F is upper weakly λ-continuous;

(ii) ClF is upper weakly λ-continuous;

(iii) $s\mathrm{Cl}\,F$ is upper weakly λ-continuous;

(iv) $\mathrm{Cl}_\lambda F$ is upper weakly λ-continuous;

(v) $\alpha\mathrm{Cl}F$ is upper weakly λ-continuous;

(vi) $p\mathrm{Cl}F$ is upper weakly λ-continuous.

Proof: We put G = ClF, $s\mathrm{Cl}\,F, \mathrm{Cl}_\lambda F, \alpha\mathrm{Cl}F$ or $p\mathrm{Cl}F$ in the sequel. Necessity: Suppose that F is upper weakly λ-continuous. Then it follows by Theorem 2.2 and Lemmas 2.12 and 2.13 that for every open set V of Y containing $F(x)$, $G^+(V) = F^+(V) \subset \mathrm{Int}_\lambda(F^+(\mathrm{Cl}(V))) = \mathrm{Int}_\lambda(G^+(\mathrm{Cl}(V)))$. By Theorem 2.2, G is upper weakly λ-continuous.

Sufficiency: Suppose that G is upper weakly λ-continuous. Then it follows by Theorem 2.2 and Lemmas 2.12 and 2.13 that for every open set V of Y containing $G(x)$, $F^+(V) = G^+(V) \subset Int_\lambda(G^+(Cl(V))) = Int_\lambda(F^+(Cl(V)))$. It follows by Theorem 2.2 that F is upper weakly λ-continuous.

Definition 2.15: Let A be a subset of a topological space X. The λ-frontier [3] of A, denoted by $Fr_\lambda(A)$, is defined by $Fr_\lambda(A) = Cl_\lambda(A) \cap Cl_\lambda(X - A)$.

Theorem 2.16: Let $F: X \to Y$ be a multifunction. The set of all points x of X such that F is not upper weakly λ-continuous (resp. lower weakly λ-continuous) is identical with the union of λ-frontiers of the upper (resp. lower) inverse images of the closure of open sets containing (resp. meetings) $F(x)$.

Proof: Let x be a point of X at which F is not upper weakly λ-continuous. Then there exists an open set V containing $F(x)$ such that $U \cap (X - F^+(Cl(V))) \neq \phi$ for every λ-open set U containing x. Therefore, $x \in Cl_\lambda(X - F^+(Cl(V)))$. Since $x \in F^+(V)$, we have $x \in Cl_\lambda(F^+(Cl(V)))$ and hence $x \in Fr_\lambda(F^+(Cl(V)))$. Conversely, if F is upper weakly λ-continuous at x, then for every open set V containing $F(x)$, there exists a λ-open set U containing x such that $F(U) \subset Cl(V)$ hence $U \subset F^+(Cl(V))$. Therefore, we obtain $x \in U \subset Int_\lambda(F^+(Cl(V)))$. This contradicts that $x \in Int_\lambda(F^+(Cl(V)))$.

Theorem 2.17: Let F and G be respectively upper weakly λ-continuous and upper weakly continuous multifunctions from a topological space (X, τ) to a strongly normal space (Y, σ). Then the set $K = \{x : F(x) \cap G(x) \neq \phi\}$ is λ-closed in X.

Proof: Let $x \in X - K$, then $F(x) \cap G(x) = \phi$. Since F and G are point closed multifunctions and Y is a strongly normal space, there exist disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively we have $Cl(U) \cap Cl(V) = \phi$. Since F and G are upper weakly λ-continuous functions, then there exist λ-open set U_1 containing x and open set U_2 containing x such that

$F(U_1) \subset Cl(V)$ and $F(U_2) \subset Cl(V)$. Now set $H = U_1 \cap U_2$, then H is an λ-open set containing x and $H \cap K = \phi$; hence K is λ-closed in X.

Definition 2.18: For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the graph multifunction of F and is denoted by $G_F(x)$.

Lemma 2.19: For a multifunction $F : X \to Y$, the following holds:

(i) $G_F{}^{+}(A \times B) = A \cap F^{+}(B)$;

(ii) $G_F{}^{-}(A \times B) = A \cap F^{-}(B)$ for any subset A of X and B of Y.

Theorem 2.20: Let $F : X \to Y$ be a multifunction and X be a connected space. If the graph multifunction of F is upper (lower) weakly λ-continuous, then F is upper (lower) weakly λ-continuous.

Proof: Let $x \in X$ and V be any open subset of Y containing $F(x)$. Since $X \times V$ is an open set relative to $X \times Y$ and $G_F(x) \subset X \times V$, there exists a λ-open set U containing x such that $G_F(U) \subset Cl(X \times V) = X \times Cl(V)$. By Lemma 2.19, we have $U \subset G^{+}{}_F(X \times Cl(V)) = F^{+}(Cl(V))$ and $F(U) \subset Cl(V)$. Thus, F is upper weakly λ-continuous. The proof of the lower weakly λ-continuity of F can be done by the same token.

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