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WAVE EQUATION WITH THREE-INVERSE SQUARE POTENTIAL ON ${R^*_+}^3$

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Abstract In this note we give explicit solutions to the wave equation associated to the Schrödinger operator with three-inverse square potential on R^{+3}.
Key words : Three inverse square potential, Cauchy problem, Wave equation, Lauricella hypergeometric functions.

1 Introduction and statement of results

The wave equation, the heat equation and the Laplace equation are known as three fundamental equations in partial differential equations and occur in many branches of physics, in applied mathematics and in engineering. In this note we give explicit formulas for the solutions of the following Cauchy problem for the wave equation with three-inverse square potential

$$(W)_{(\nu,\nu',\nu'')}\left\{\begin{array}{llr}(a) & [\Delta+v_{\nu,\nu',\nu''}(x)]u(t,p)=\frac{\partial^2}{\partial t^2}u(t,p) & (t,p)\in R\times R^3_+\\ (b) & u(0,p)=0 & \frac{\partial}{\partial t}u(0,p)=u_1(p), u_1\in C_0^\infty(R^3_+)\end{array}\right.$$

with $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacien of R^3, the three inverse square potential is given by $v_{\nu,\nu',\nu''}(x,y,z)=\frac{1/4-\nu^2}{x^2}+\frac{1/4-\nu'^2}{y^2}+\frac{1/4-\nu''^2}{z^2}$ where ν,ν',ν'' are real parameters. The Cauchy problem for the wave equation with the inverse square potential in Euclidean space $\mathbb{R}^n$ is extensively studied (Cheeger and Taylor [3])). The bi-inverse square potential has been considered by (Boyer [2])and Ould Moustapha [7]. The case considered most frequentely is obviously the one where $(\nu,\nu',\nu'')=(\pm1/2,\pm1/2,\pm1/2)$, the equation in $(W)_{\nu,\nu',\nu''}$ then turns into the classical wave equation on the Euclidean space $\mathbb{R}^3$ and this equation appears in several branches of mathematics and physics (Folland $[5], p.171$). Now we state the main results of this paper:

Theorem A *For* $(t,p,p')\in R_+\times{\mathbb{R}^*}^3_+\times{\mathbb{R}^*}^3_+$ *the functions:*
$W_{(b,b',b'')}(t,p,p')=\frac{c_3(xx')^b(yy')^{b'}(zz')^{b''}}{(t^2-|p+p'|^2)^{1+b+b'+b''}}F_A^{(3)}\left(1+b+b'+b'',b,b',b'',2b,2b',2b'',\frac{-4xx'}{t^2-|p+p'|^2},\frac{-4yy'}{t^2-|p+p'|^2},\frac{-4zz'}{t^2-|p+p'|^2}\right)$
with $b\in\{\beta,1-\beta\}$, $b'\in\{\beta',1-\beta'\}$, $b''\in\{\beta'',1-\beta''\}$ *are independent solutions of the wave equation with three-inverse square potential on* $\mathbb{R}^3_+$ (a) *where* $\beta=1/2+\nu$ *and* $\beta'=1/2+\nu'$, $\beta''=1/2+\nu''$ *and* $F_A^{(3)}(a,b,b',b'',c,c',c'';w,w',w'')$ *is the three variables triple series* $F_A^{(3)}$ *Lauricella hypergeometric function given by* $[1], p.114$

$$F_A^{(3)}\left(a,b,b',b'',c,c',c'',w,w',w''\right)=\sum_{m,n,p\geq0}\frac{(a)_{m+n+p}(b)_m(b')_n(b'')_p}{(c)_m m!(c')_n n!(c'')_p p!}w^m w'^n w''^p \quad (1.1)$$

Theorem B *The Cauchy problem for the wave equation with three-inverse square potential on the* $\mathbb{R}^3_+$ *has the solutions given by:*

$$u(t,p)=\frac{\partial}{t\partial t}\int_{|p+p'|<t}W_{(b,b',b'')}(t,p,p')f(p')dp' \quad (1.2)$$

where the kernel $W_{(b,b',b'')}$ *is as in the Theorem A and the constant* c_3 *is given by*

$$c_3=\frac{(-1)^{b+b'+b''}\Gamma(1/2+b+b'+b'')}{\sqrt{\pi}\Gamma(1/2+b)\Gamma(1/2+b')\Gamma(1/2+b'')} \quad (1.3)$$

with $dp'=dx'dy'dz'$ *is the Lebesgue measure on* $\mathbb{R}^3$

2 Wave equation with three-inverse square potential on Euclidean space $\mathbb{R}^3$

Proof of Theorem A *In what follows we give a direct proof of the theorem A.*
Let $a = t^2 - |p+p'|^2$, $t \in \mathbb{R}$, $p, p' \in \mathbb{R}^{*3}$ *set:*
$\Omega\varphi(t,p) = (xx')^{-\beta}(yy')^{-\beta'}(zz')^{-\beta''}a^{-\alpha}\times$

$$\left[\Delta - \frac{\beta(\beta-1)}{x^2} - \frac{\beta'(\beta'-1)}{y^2} - \frac{\beta''(\beta''-1)}{z^2} - \frac{\partial^2}{\partial t^2}\right](xx')^{\beta}(yy')^{\beta'}(zz')^{\beta''}a^{\alpha}\varphi(t,x) \tag{2.1}$$

then we have
$\Omega\varphi(t,p) = \{\Delta - \frac{\partial^2}{\partial t^2} + [\frac{2\beta}{x} - \frac{4\alpha(x+x')}{a}]\frac{\partial}{\partial x} + [\frac{2\beta'}{y} - \frac{4\alpha(y+y')}{a}]\frac{\partial}{\partial y} + [\frac{2\beta''}{z} - \frac{4\alpha(z+z')}{a}]\frac{\partial}{\partial z} - \frac{4\alpha t}{a}\frac{\partial}{\partial t}$

$$-\frac{4\alpha}{a}[\frac{\beta x'}{x} + \frac{\beta' y'}{y} + \frac{\beta'' z'}{z}] - \frac{4\alpha}{a}[\alpha + 1 + \beta + \beta' + \beta'']\}\varphi(t,x) \tag{2.2}$$

Now set

$$w = \frac{-4xx'}{t^2 - |p+p'|^2}, w' = \frac{-4yy'}{t^2 - |p+p'|^2}, w'' = \frac{-4zz'}{t^2 - |p+p'|^2} \tag{2.3}$$

we can write:

$$\frac{\partial}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial}{\partial w} + \frac{\partial w'}{\partial x}\frac{\partial}{\partial w'} + \frac{\partial w''}{\partial x}\frac{\partial}{\partial w''}; \frac{\partial}{\partial y} = \frac{\partial w}{\partial y}\frac{\partial}{\partial w} + \frac{\partial w'}{\partial y}\frac{\partial}{\partial w'} + \frac{\partial w''}{\partial y}\frac{\partial}{\partial w''} \tag{2.4}$$

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} + \frac{\partial w'}{\partial z}\frac{\partial}{\partial w'} + \frac{\partial w''}{\partial z}\frac{\partial}{\partial w''}; \frac{\partial}{\partial t} = \frac{\partial w}{\partial t}\frac{\partial}{\partial w} + \frac{\partial w'}{\partial t}\frac{\partial}{\partial w'} + \frac{\partial w''}{\partial t}\frac{\partial}{\partial w''} \tag{2.5}$$

We have

$$\begin{aligned}
\Omega\varphi(t,p) &= \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 - \left(\frac{\partial w}{\partial t}\right)^2\right]\frac{\partial^2}{\partial w^2} \\
&+\left[\left(\frac{\partial w'}{\partial x}\right)^2 + \left(\frac{\partial w'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2 - \left(\frac{\partial w'}{\partial t}\right)^2\right]\frac{\partial^2}{\partial w'^2} + \left[\left(\frac{\partial w''}{\partial x}\right)^2 + \left(\frac{\partial w''}{\partial y}\right)^2 + \left(\frac{\partial w''}{\partial z}\right)^2 - \left(\frac{\partial w''}{\partial t}\right)^2\right]\frac{\partial^2}{\partial w''^2} \\
&+2\left[\frac{\partial w}{\partial x}\frac{\partial w'}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w'}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w'}{\partial z} - \frac{\partial w}{\partial t}\frac{\partial w'}{\partial t}\right]\frac{\partial^2}{\partial w\partial w'} + 2\left[\frac{\partial w}{\partial x}\frac{\partial w''}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w''}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w''}{\partial z} - \frac{\partial w}{\partial t}\frac{\partial w''}{\partial t}\right]\frac{\partial^2}{\partial w\partial w''} \\
&+ 2\left[\frac{\partial w'}{\partial x}\frac{\partial w''}{\partial x} + \frac{\partial w'}{\partial y}\frac{\partial w''}{\partial y} + \frac{\partial w'}{\partial z}\frac{\partial w''}{\partial z} - \frac{\partial w'}{\partial t}\frac{\partial w''}{\partial t}\right]\frac{\partial^2}{\partial w'\partial w''} + \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial t^2}\right]\frac{\partial}{\partial w} \\
&+ [\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} - \frac{\partial^2 w'}{\partial t^2}]\frac{\partial}{\partial w'} + [\frac{\partial^2 w''}{\partial x^2} + \frac{\partial^2 w''}{\partial y^2} + \frac{\partial^2 w''}{\partial z^2} - \frac{\partial^2 w''}{\partial t^2}]\frac{\partial}{\partial w''} \\
&+ [A_x\frac{\partial w}{\partial x} + A_y\frac{\partial w}{\partial y} + A_z\frac{\partial w}{\partial z} - A_t\frac{\partial w}{\partial t}]\frac{\partial}{\partial w} + [A_x\frac{\partial w'}{\partial x} + A_y\frac{\partial w'}{\partial y} + A_z\frac{\partial w'}{\partial z} - A_t\frac{\partial w'}{\partial t}]\frac{\partial}{\partial w'} \\
&+[A_x\frac{\partial w''}{\partial x} + A_y\frac{\partial w''}{\partial y} + A_z\frac{\partial w''}{\partial z} - A_t\frac{\partial w''}{\partial t}]\frac{\partial}{\partial w''} + \frac{4\alpha}{a}[\frac{\beta x'}{x} + \frac{\beta' y'}{y} + \frac{\beta'' z'}{z}] + \frac{4\alpha}{a}\left[\alpha + 1 + \beta + \beta' + \beta''\right]\varphi(z,z')
\end{aligned} \tag{2.6}$$

where

$$A_x = \frac{2\beta}{x} - \frac{4\alpha(x+x')}{a}, A_y = \frac{2\beta'}{y} - \frac{4\alpha(y+y')}{a}, A_z = \frac{2\beta''}{z} - \frac{4\alpha(z+z')}{a}, A_t = \frac{4\alpha t}{a} \tag{2.7}$$

We have:

$$\frac{\partial w}{\partial x} = \frac{-4x'a - 8(x+x')xx'}{a^2}; \frac{\partial w'}{\partial x} = \frac{-8yy'(x+x')}{a^2}; \frac{\partial w''}{\partial x} = \frac{-8zz'(x+x')}{a^2} \tag{2.8}$$

$$\frac{\partial w}{\partial y} = \frac{-8xx'(y+y')}{a^2}; \frac{\partial w'}{\partial y} = \frac{-4y'a - 8yy'(y+y')}{a^2}; \frac{\partial w''}{\partial y} = \frac{-8zz'(y+y')}{a^2} \tag{2.9}$$

$$\frac{\partial w}{\partial z} = \frac{-8xx'(z+z')}{a^2}; \frac{\partial w'}{\partial z} = \frac{-8yy'(z+z')}{a^2}; \frac{\partial w''}{\partial z} = \frac{-4z'a - 8zz'(z+z')}{a^2} \tag{2.10}$$

$$\frac{\partial w}{\partial t}=\frac{8xx't}{a^2};\ \frac{\partial w'}{\partial t}=\frac{8yy't}{a^2},\ \frac{\partial w''}{\partial t}=\frac{8zz't}{a^2} \tag{2.11}$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{-24xx'a^2-16x'^2a^2-32(x+x')^2xx'a}{a^4};\ \frac{\partial^2 w}{\partial y^2}=\frac{-8yy'a^2-32xx'(y+y')^2a}{a^4};\ \frac{\partial^2 w}{\partial z^2}=\frac{-8zz'a^2-32xx'(z+z')^2a}{a^4} \tag{2.12}$$

$$\frac{\partial^2 w'}{\partial x^2}=\frac{-8xx'a^2-32yy'(x+x')^2a}{a^4};\ \frac{\partial^2 w'}{\partial y^2}=\frac{-24yy'a^2-16y'^2a^2-32(y+y')^2yy'a}{a^4};\ \frac{\partial^2 w'}{\partial z^2}=\frac{-8yy'a^2-32azz'(y+y')^2a}{a^4} \tag{2.13}$$

$$\frac{\partial^2 w''}{\partial x^2}=\frac{-8xx'a^2-32zz'(x+x')^2a}{a^4};\ \frac{\partial^2 w''}{\partial y^2}=\frac{-8yy'a^2-32zz'(y+y')^2a}{a^4};\ \frac{\partial^2 w''}{\partial z^2}=\frac{-24zz'a^2-16z'^2a^2-32(z+z')^2zz'a}{a^4} \tag{2.14}$$

$$\frac{\partial^2 w}{\partial t^2}=\frac{8xx'a^2-32axx't^2}{a^4};\ \frac{\partial^2 w'}{\partial t^2}=\frac{8yy'a^2-32ayy't^2}{a^4},\ \frac{\partial^2 w''}{\partial t^2}=\frac{8zz'a^2-32azz't^2}{a^4} \tag{2.15}$$

from $(2.8)-(2.11)$ *we have*

$$\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2-\left(\frac{\partial w}{\partial t}\right)^2=\frac{w^2}{x^2}(1-w) \tag{2.16}$$

$$\left(\frac{\partial w'}{\partial x}\right)^2+\left(\frac{\partial w'}{\partial y}\right)^2+\left(\frac{\partial w'}{\partial z}\right)^2-\left(\frac{\partial w'}{\partial t}\right)^2=\frac{w'^2}{y^2}(1-w') \tag{2.17}$$

$$\left(\frac{\partial w''}{\partial x}\right)^2+\left(\frac{\partial w''}{\partial y}\right)^2+\left(\frac{\partial w''}{\partial z}\right)^2-\left(\frac{\partial w''}{\partial t}\right)^2=\frac{w''^2}{z^2}(1-w'') \tag{2.18}$$

$$2\left[\frac{\partial w}{\partial x}\frac{\partial w'}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial w'}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial w'}{\partial z}-\frac{\partial w}{\partial t}\frac{\partial w'}{\partial t}\right]=\frac{w^2}{x^2}w'+\frac{w'^2}{y^2}w \tag{2.19}$$

$$2\left[\frac{\partial w}{\partial x}\frac{\partial w''}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial w''}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial w''}{\partial z}-\frac{\partial w}{\partial t}\frac{\partial w''}{\partial t}\right]=\frac{w^2}{x^2}w''+\frac{w''^2}{z^2}w \tag{2.20}$$

$$2\left[\frac{\partial w'}{\partial x}\frac{\partial w''}{\partial x}+\frac{\partial w'}{\partial y}\frac{\partial w''}{\partial y}+\frac{\partial w'}{\partial z}\frac{\partial w''}{\partial z}-\frac{\partial w'}{\partial t}\frac{\partial w''}{\partial t}\right]=\frac{w'^2}{y^2}w''+\frac{w''^2}{z^2}w' \tag{2.21}$$

from $(2.12)-(2.15)$ *we have*

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}-\frac{\partial^2 w}{\partial t^2}=\frac{-w^2}{x^2}+\frac{2}{a}w \tag{2.22}$$

$$\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}+\frac{\partial^2 w'}{\partial z^2}-\frac{\partial^2 w'}{\partial t^2}=\frac{-w'^2}{y^2}+\frac{2}{a}w' \tag{2.23}$$

$$\frac{\partial^2 w''}{\partial x^2}+\frac{\partial^2 w''}{\partial y^2}+\frac{\partial^2 w''}{\partial z^2}-\frac{\partial^2 w''}{\partial t^2}=\frac{-w''^2}{z^2}+\frac{2}{a}w'' \tag{2.24}$$

from $(2.7)-(2.11)$ *we have*

$$A_x\frac{\partial w}{\partial x}+A_y\frac{\partial w}{\partial y}+A_z\frac{\partial w}{\partial z}-A_t\frac{\partial w}{\partial t}=\frac{4\alpha+4(\beta+\beta'+\beta'')}{a}w+2\beta\frac{w}{x^2}+\alpha\frac{w^2}{x^2}-\beta\frac{w^2}{x^2}+(\frac{\beta' w'}{y^2}+\frac{\beta'' w''}{z^2})w \tag{2.25}$$

$$A_x\frac{\partial w'}{\partial x}+A_y\frac{\partial w'}{\partial y}+A_z\frac{\partial w'}{\partial z}-A_t\frac{\partial w'}{\partial t}=\frac{4\alpha+4(\beta+\beta'+\beta'')}{a}w'+2\beta\frac{w'}{y^2}+\alpha\frac{w'^2}{y^2}-\beta\frac{w'^2}{y^2}+(\frac{\beta w}{x^2}+\frac{\beta'' w''}{z^2})w' \tag{2.26}$$

$$A_x\frac{\partial w''}{\partial x}+A_y\frac{\partial w''}{\partial y}+A_z\frac{\partial w''}{\partial z}-A_t\frac{\partial w''}{\partial t}=\frac{4\alpha+4(\beta+\beta'+\beta'')}{a}w''+2\beta''\frac{w''}{z^2}+\alpha\frac{w''^2}{z^2}-\beta''\frac{w''^2}{z^2}+(\frac{\beta w}{x^2}+\frac{\beta' w'}{y^2})w'' \tag{2.27}$$

To replace in the formula (2.6) *using the formulas* $(2.16)-(2.27)$ *we get:*
$\Omega\varphi=wx^{-2}A_{\alpha,\beta}(w,w',w'')\varphi+w'y^{-2}A_{\alpha,\beta'}(w',w,w'')\varphi+w''z^{-2}A_{\alpha,\beta''}(w'',w,w')\varphi+$

$$\frac{4}{a}(\alpha+\beta+\beta'+\beta''+1)\left[\left(w\frac{\partial}{\partial w}+w'\frac{\partial}{\partial w'}+w''\frac{\partial}{\partial w''}\right)\varphi(w,w',w'')-\varphi(w,w',w'')\right] \tag{2.28}$$

Take $\alpha = -1-\beta-\beta'-\beta''$ *we get* $\Omega\varphi = 0$ *is equivalent to*

$$wx^{-2}A_{\alpha,\beta}(w,w',w'')\varphi + w'y^{-2}A_{\alpha,\beta'}(w',w,w'')\varphi + w''z^{-2}A_{\alpha,\beta''}(w'',w,w')\varphi = 0 \tag{2.29}$$

with
$A_{\alpha,\beta}(w,w',w'')\varphi(w,w',w'') = [w(1-w)\frac{\partial^2}{\partial w^2} - w(w'\frac{\partial^2}{\partial w'\partial w} + w''\frac{\partial^2}{\partial w''\partial w})+$

$$+[2\beta + (-\alpha+\beta+1)w]\,\frac{\partial}{\partial w} - \beta' w'\frac{\partial}{\partial w'} - \beta'' w''\frac{\partial}{\partial w''} + \alpha\beta]\varphi(w,w',w'') \tag{2.30}$$

From the formula (2.29) *we have*

$$A_{\alpha,\beta}(w,w',w'')\varphi(w,w',w'') = A_{\alpha,\beta'}(w',w,w'')\varphi(w,w',w'') = A_{\alpha,\beta''}(w,w',w)\varphi(w,w',w'') = 0 \tag{2.31}$$

$$[w(1-w)\frac{\partial^2}{\partial w^2} - w(w'\frac{\partial^2}{\partial w'\partial w} + w''\frac{\partial^2}{\partial w''\partial w}) + [2\beta+(-\alpha+\beta+1)w]\,\frac{\partial}{\partial w} - \beta'w'\frac{\partial}{\partial w'} - \beta''w''\frac{\partial}{\partial w''} + \alpha\beta]\varphi(w,w',w'') = 0,$$
$$[w'(1-w')\frac{\partial^2}{\partial w'^2} - w'(w\frac{\partial^2}{\partial w\partial w'} + w''\frac{\partial^2}{\partial w''\partial w'}) + \left[2\beta'+(-\alpha+\beta'+1)w'\right]\,\frac{\partial}{\partial w'} - \beta w\frac{\partial}{\partial w} - \beta''w''\frac{\partial}{\partial w''} + \alpha\beta']\varphi(w,w',w'') = 0,$$
$$[w''(1-w'')\frac{\partial^2}{\partial w''^2} - w''(w'\frac{\partial^2}{\partial w''\partial w'} + w\frac{\partial^2}{\partial w''\partial w}) + \left[2\beta''+(-\alpha+\beta''+1)w''\right]\,\frac{\partial}{\partial w''} - \beta w\frac{\partial}{\partial w} - \beta'w'\frac{\partial}{\partial w'} + \alpha\beta'']\varphi(w,w',w'') = 0,$$

This is an $F_A^{(3)}$ *three variable Laurichella hypergeometric system and for* $2\beta \neq 1$ *and* $2\beta' \neq 1$ $2\beta'' \neq 1$ *the system has six independent solutions of the form* $[4], p.150-151$:

- $F_A^{(3)}(-\alpha,\beta,\beta',\beta'',2\beta,2\beta',2\beta'',w,w',w'')$,
- $w^{1-2\beta}F_A^{(3)}(-\alpha+1-2\beta,1-\beta,\beta',\beta'',2-2\beta,2\beta',2\beta'',w,w',w'')$,
- $w'^{1-2\beta'}F_A^{(3)}(-\alpha+1-2\beta',\beta,1-\beta',\beta'',2\beta,1-2\beta',2\beta'',w,w',w'')$,
- $w''^{1-2\beta''}F_A^{(3)}(-\alpha+1-2\beta'',\beta,\beta',1-\beta'',2\beta,2\beta',2-2\beta'',w,w',w'')$,
- $w^{1-2\beta}w'^{1-2\beta'}F_A^{(3)}(-\alpha+2-2\beta-2\beta',1-\beta,1-\beta',\beta'',2-2\beta,2-2\beta',2\beta'',w,w',w'')$,
- $w^{1-2\beta}w''^{1-2\beta''}F_A^{(3)}(-\alpha+2-2\beta-2\beta'',1-\beta,\beta',1-\beta'',2-2\beta,2\beta',2-2\beta'',w,w',w'')$,
- $w'^{1-2\beta}w''^{1-2\beta}F_A^{(3)}(-\alpha+2-2\beta'-2\beta'',\beta,1-\beta',1-\beta'',2\beta,2-2\beta',2-2\beta'',w,w',w'')$,
- $w^{1-2\beta}w'^{1-2\beta}w''^{1-2\beta}F_A^{(3)}(-\alpha+3-2\beta-2\beta'-2\beta'',1-\beta,1-\beta',1-\beta'',2-2\beta,2-2\beta',2-2\beta'',w,w',w'')$,
  And the proof of the theorem 1.1 is finished.

3 Cauchy problem for the wave equation with the three-inverse square potential on $\mathbf{R}^{+3}$

Proof of the Theorem B
Lemma 3.1 *Let* $F_A^{(3)}$ *be the Appell hypergeometric function with* $(h,k,l) \in R^3$ *and* $a \in R^*$ *then we have:*

i) $\frac{d}{da}\left[a^{\alpha}F_A^{(3)}\left(-\alpha,\beta,\beta',\beta'',2\beta,2\beta',2\beta'',h/a,k/a,l/a\right)\right] = -\alpha a^{\alpha-1}\times$

$$F_A^{(3)}\left(-\alpha+1,\beta,\beta',\beta'',2\beta,2\beta',2\beta'',h/a,k/a,l/a\right) \tag{3.1}$$

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ii) $a^{\alpha}\Gamma(-\alpha)F_A^{(3)}(-\alpha,\beta,\beta',\beta'',2\beta,2\beta',2\beta'',h/a,k/a,l/a) = \Gamma(\beta+1/2)\Gamma(\beta'+1/2)\Gamma(\beta''+1/2)\times$ $\int_0^\infty e^{-(a-h/2-k/2-l/2)t}t^{-\alpha-1}\left(\frac{th}{4}\right)^{1/2-\beta}\left(\frac{tk}{4}\right)^{1/2-\beta'}\left(\frac{tk}{4}\right)^{1/2-\beta''}\times$

$$I_{\beta-1/2}(th/2)I_{\beta'-1/2}(tk/2)I_{\beta''-1/2}(tl/2)dt, \tag{3.2}$$

iii) $a^{\alpha}\Gamma(-\alpha)F_A^{(3)}(-\alpha,\beta,\beta',\beta''\gamma,\gamma',\gamma'',h/a,k/a,l/a)\sim$

$$\Gamma(-\beta-\beta'-\beta''-\alpha)\frac{\Gamma(2\beta)}{\Gamma(\beta)}\frac{\Gamma(2\beta')}{\Gamma(\beta')}\frac{\Gamma(2\beta'')}{\Gamma(\beta'')}h^{-\beta}k^{-\beta'}l^{-\beta''}(a-h-k-l)_+^{\alpha+\beta+\beta'+\beta''} \tag{3.3}$$

as $a\to 0$.

.

Proof

i) is a consequence of the formulas
$a^{\alpha}F_A^{(3)}(-\alpha,\beta,\beta',\beta'',\gamma,\gamma',\gamma'',h/a,k/a,l/a) =$

$$\sum_{m,n,p\geq 0}\frac{(-\alpha)_{m+n+p}(\beta)_m(\beta')_n(\beta'')_p}{(\gamma)_m m!(\gamma')_n n!(\gamma'')_p p!}h^m k^n l^p a^{-m-n-p+\alpha} \tag{3.4}$$

$\frac{d}{da}[a^{\alpha}F_A^{(3)}](-\alpha,\beta,\beta',\beta'',\gamma,\gamma',\gamma'',h/a,k/a,l/h)] = -\alpha a^{\alpha-1}\times$

$$\sum_{m,n,p\geq 0}\frac{(-\alpha+1)_{m+n+p}(\beta)_m(\beta')_n(\beta'')_p}{(\gamma)_m(\gamma')_n(\gamma'')_p m!n!p!}h^m k^n l^p a^{-m-n-p} \tag{3.5}$$

$\frac{d}{da}[a^{\alpha}F_A^{(3)}](-\alpha,\beta,\beta',\gamma,\gamma',\gamma'',h/a,k/a,l/a)] = -\alpha a^{\alpha-1}\times$

$$F_A\left(-\alpha+1,\beta,\beta',\beta'',\gamma,\gamma',\gamma'',h/a,k/a,l/a\right) \tag{3.6}$$

To prove ii) we use the formulas([8]*, p.237)*

$$I_\nu(z) = \frac{2z)^\nu e^{-z}}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^1 e^{2zt}\,[t(1-t)]^{\nu-1/2}\,dt \tag{3.7}$$

where $\alpha\in C,\beta,\gamma,z\in R^n$ [1], p.115

$$F_A^{(3)}(\alpha,\beta,\gamma,z) = c\int_0^1\int_0^1\int_0^1\prod_{j=1}^{j=3}(1-u_j)^{\gamma_j-\beta_j-1}u_j^{\beta_j-1}\left(1-\sum_{j=1}^{j=3}u_jz_j\right)^{-\alpha}du_j \tag{3.8}$$

where

$$c = \frac{\prod_{j=1}^{j=3}\Gamma(\gamma_j)}{\prod_{j=1}^{j=3}\Gamma(\beta_j)\Gamma(\gamma_j-\beta_j)} \tag{3.9}$$

iii) The Proof of iii) uses ii) and the formula ([8]*, p.240)*

$$I_{\beta_j-1/2}(x) = \frac{\Gamma(2\beta_j)}{\Gamma(\beta_j)\Gamma(\beta_j+1/2)}2^{-2\beta_j+1/2}x^{-1/2}\left(1+O(|x|^{-1})\right) \tag{3.10}$$

To finish the proof of the theorem 1.2, we prove the limit conditions in (b)*: from iii) of the Lemma 2.1 and the Legendre duplication formula for the* Γ*-Euler function* [6], p.3

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z). \tag{3.11}$$

we have For $t\longrightarrow 0$ *and* $p,p'\in\mathbb{R}_+^{*\,3}$

$$W_{(b,b',b'')}^{\mathbb{R}^3}(t,p,p')\sim c_3 1_{\{|p-p'|<t\}}. \tag{3.12}$$

The polar coordinates $p' = p + r\omega$ *for* $t \longrightarrow 0$

$$u(t,p) \sim c_3 \frac{\partial}{t\partial t} \int_{\mathbb{R}^{*3}} 1_{\{|p-p'|<t\}} f(p')dp' \tag{3.13}$$

$$u(t,p) \sim c_3 \frac{\partial}{t\partial t} \int_0^t f_p^{\#}(r) r^2 dr \tag{3.14}$$

$$f_p^{\#}(r) = \frac{1}{6} \int_{S^2} f(p + r\omega) d\omega \tag{3.15}$$

set $r = st$ *in the expression above and we see that the limit condition* (b) *is satisfied and the proof of the theorem B is finished.*

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