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EXISTENCE OF COMMON FIXED POINT IN COMPLETE METRIC SPACES

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Abstract. In the present paper we prove a common fixed point theorem for pairs of mappings on complete metric spaces by assuming that one of given mapping is continuous. Our results generalize and extend some recently announced results in the literature.

1. introduction

Fixed point theory for continuous and related mappings has played a very important role in many aspects of nonlinear functional analysis for many years. Basic idea in this paper is, to the study of fixed points of some mappings in complete metric spaces. we improve and extend some results due to RK Namdeo *et al* [6], P.P. Muthy *et al* [1]. Some related fixed point theorem on two and more complete metric spaces also studied by V. Popa [8], S.C. Nesic [7], Vishal Gupta [2]- [5]. In this paper, we prove a related fixed point theorem for two mappings, by assuming that one of given mapping is continuous, on two metric spaces. Thus our theorem improves theorem (2.1) and (2.2).

2. Preliminaries

Definition 2.1. *Let (X, d) be metric space. A sequence $\{x_n\} \in X$ is said to be converge to a point $p \in X \iff \forall\, \varepsilon > 0\ \exists$ a positive integer $n_0(\varepsilon)$ such that*

$$d\,(x_n, p) < \varepsilon,\ \forall\ n \geqslant n_0$$

Definition 2.2. *Let (X, d) be a metric space, a sequence $\{x_n\} \in X$ is said to be Cauchy sequence if $d(x_m, x_n) \to 0$ as $m, n \to \infty$.*

Definition 2.3. *A metric space (X, d) is said to be complete iff every Cauchy sequence in X converges to a point of X.*

The following fixed point theorem was proved by B Fisher *et al* [1].

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Theorem 2.1. *Let (X, d) and (Y, ρ) be complete metric spaces and let $A, B : X \to Y$ and $S, T : Y \to X$ satisfying the inequalities.*

$$
\begin{aligned}
d\left(SAx, TBx'\right) &\leqslant c \max\{d(x, x'), d(x, SAx)\\
&\qquad d\left(x', TBx'\right), \rho\left(Ax, Bx'\right)\}\\
\rho\left(BSy, ATy'\right) &\leqslant c \max\left\{\rho\left(y, y'\right), \rho\left(y, BSy\right),\right.\\
&\qquad \left.\rho\left(y', ATy'\right), d\left(Sy, Ty'\right)\right\}
\end{aligned}
$$

$\forall\ x, x' \in X$ and $y, y' \in Y$ where $0 \leqslant c < 1$. If one of the mappings A, B, S and T is continuous. The SA and TB have a unique common fixed point $z \in X$ and BS and AT have a unique fixed point $w \in Y$. Further $Az = Bz = w$ and $Sw = Tw = z$.

The next theorem was proved by RK Namdeo *et al* [6].

Theorem 2.2. *Let (X, d) and (y, ρ) be complete metric spaces. Let $T : X \to Y$ and $S : Y \to X$ satistying the inequalities.*

$$
\begin{aligned}
d\left(Sy, Sy'\right) d\left(STx, STx'\right) &\leqslant c \max\{d(Sy, Sy')\rho(Tx, Tx'),\\
&\qquad d\left(x', Sy\right) \rho\left(y', Tx\right),\\
&\qquad d\left(x, x'\right) d\left(Sy, Sy'\right),\\
&\qquad d\left(Sy, STx\right) d\left(Sy', STx'\right)\}\\
\rho\left(Tx, Tx'\right)\left(TSy, TSy'\right) &\leqslant c \max\{d\left(Sy, Sy'\right) \rho\left(Tx, Tx'\right)\\
&\qquad d\left(x', Sy\right) \rho\left(y', Tx\right),\\
&\qquad \rho\left(y, y'\right) \rho\left(Tx, Tx'\right),\\
&\qquad \rho\left(Tx, TSy\right) \rho\left(Tx', TSy'\right)\}
\end{aligned}
$$

$\forall\ x, x' \in X$ and $y, y' \in Y$, where $0 \leqslant c < 1$ If either T or S is continuous then ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further $Tz = w$ and $Sw = z$.

3. Main Result

We now prove the following related fixed point theorem.

Theorem 3.1. *Let (X, d_1) and (Y, d_2) be complete metric spaces. Let $A, B : X \to Y$ and $S, T : Y \to X$ satistying the inequalities.*

$$\begin{aligned} d_1(Sy, Ty')\, d_1(SAx, TBx') \leqslant c \max\{&d_1(Sy, Ty') d_2(Ax, Bx'), \\ &d_1(x', Sy)\, d_2(y', Ax), \\ &d_1(x, x')\, d_1(Sy, Ty'), \\ &d_1(Sy, SAx)\, d_1(Ty', TBx')\} \end{aligned} \quad (3.1)$$

$$\begin{aligned} d_2(Ax, Bx')\, d_2(BSy, ATy') \leqslant c \max\{&d_1(Sy, Ty')\, d_2(Ax, Bx') \\ &d_1(x', Sy)\, d_2(y', Ax), \\ &d_2(y, y')\, d_2(Ax, Bx'), \\ &d_2(Ax, BSy)\, d_2(Bx', ATy')\} \end{aligned} \quad (3.2)$$

$\forall\ x, x' \in X$ and $y, y' \in Y$, where $0 \leqslant c < 1$. If one of the mappings A, B, S, T is continuous then SA and TB have a unique common fixed point $z \in X$ and BS and AT have unique common fixed point $w \in Y$. Further $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X. Let,

$$Ax = y_1, \quad Sy_1 = x_1, \quad Bx_1 = y_2, \quad Ty_2 = x_2, \quad Ax_2 = y_3$$

and in general let,

$$\begin{aligned} Sy_{2n-1} &= x_{2n-1}, & Bx_{2n-1} &= y_{2n}, \\ Ty_{2n} &= x_{2n}, & Ax_{2n} &= y_{2n+1} \end{aligned}$$

For $n = 1, 2, \ldots$ using inequality (3.1), we get,

$$\begin{aligned} d_1(x_{2n-1}, x_{2n})\, d_1(x_{2n}, x_{2n+1}) &= d_1(Sy_{2n-1}, Ty_{2n})\, d_1(SAx_{2n-1}, TBx_{2n}) \\ &\leqslant c \max \\ &\quad \{d_1(Sy_{2n-1}, Ty_{2n})\, d_2(Ax_{2n-1}, Bx_{2n}), \\ &\quad d_1(x_{2n}, Sy_{2n-1})\, d_2(y_{2n}, Ax_{2n-1}), \\ &\quad d_1(x_{2n-1}, x_{2n})\, d_1(Sy_{2n-1}, Ty_{2n}), \\ &\quad d_1(Sy_{2n-1}, SAx_{2n-1})\, d_1(Ty_{2n}, TBx_{2n})\} \\ &\leqslant c \max\{d_1(x_{2n-1}, x_{2n})\, d_2(y_{2n}, y_{2n+1}), \\ &\quad d_1(x_{2n}, x_{2n-1})\, d_2(y_{2n}, y_{2n}), \\ &\quad d_1(x_{2n-1}, x_{2n})\, d_1(x_{2n-1}, x_{2n}), \\ &\quad d_1(x_{2n-1}, x_{2n})\, d_1(x_{2n}, x_{2n+1})\} \\ d_1(x_{2n-1}, x_{2n})\, d_1(x_{2n}, x_{2n+1}) &\leqslant c \max\{d_1(x_{2n-1}, x_{2n})\, d_2(y_{2n}, y_{2n+1}), \\ &\quad [d_1(x_{2n-1}, x_{2n})]^2, \end{aligned}$$

$$d_1(x_{2n-1}, x_{2n})\, d_1(x_{2n}, x_{2n+1})\}$$

From which it follows that

$$d_1(x_{2n}, x_{2n+1}) \leqslant c \max\{d_2(y_{2n}, y_{2n+1}), d_1(x_{2n-1}, x_{2n})\} \tag{3.3}$$

Now using inequality (3.1) again, it follows similarly that

$$d_1(x_{2n-1}, x_{2n}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n-1}), d_2(y_{2n}, y_{2n-1})\} \tag{3.4}$$

Using inequality (3.2), we have,

$$\begin{aligned}[d_2(y_{2n}, y_{2n+1})]^2 &= d_2(Ax_{2n-1}, Bx_{2n})\, d_2(BSy_{2n-1}, ATy_{2n}) \\ &\leqslant c \max\{d_1(Sy_{2n-1}, Ty_{2n})\, d_2(Ax_{2n-1}, Bx_{2n}), \\ &\quad d_1(x_{2n}, Sy_{2n-1})\, d_2(y_{2n}, Ax_{2n-1}), \\ &\quad d_2(y_{2n-1}, y_{2n})\, d_2(Ax_{2n-1}, Bx_{2n}), \\ &\quad d_2(Ax_{2n-1}, BSy_{2n-1})\, d_2(Bx_{2n}, ATy_{2n})\} \\ &\leqslant c \max\{d_1(x_{2n-1}, x_{2n})\, d_2(y_{2n}, y_{2n+1}), \\ &\quad d_1(x_{2n}, x_{2n-1})\, d_2(y_{2n}, y_{2n}), \\ &\quad d_2(y_{2n-1}, y_{2n})\, d_2(y_{2n}, y_{2n+1}), \\ &\quad d_2(y_{2n}, Bx_{2n-1})\, d_2(y_{2n+1}, Ax_{2n})\}\end{aligned}$$

From which it follows that,

$$d_2(y_{2n}, y_{2n+1}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n}), d_2(y_{2n-1}, y_{2n})\} \tag{3.5}$$

$$d_2(y_{2n-1}, y_{2n}) \leqslant c \max\{d_1(x_{2n-2}, x_{2n-1}), d_2(y_{2n-2}, y_{2n-1})\} \tag{3.6}$$

Using inequalities (3.3) and (3.5), we have,

$$\begin{aligned}d_1(x_{2n+1}, x_{2n}) &\leqslant c \max\{d_1(x_{2n}, x_{2n-1}), d_2(y_{2n}, y_{2n+1})\} \\ &\leqslant c \max\{d_1(x_{2n}, x_{2n-1}), \\ &\quad cd_1(x_{2n-1}, x_{2n}), cd_2(y_{2n-1}, y_{2n})\}\end{aligned}$$

$$d_1(x_{2n+1}, x_{2n}) \leqslant c \max\{d_1(x_{2n}, x_{2n-1}), d_2(y_{2n-1}, y_{2n})\} \tag{3.7}$$

Similarly from inequalities (3.4) and (3.6), we have,

$$d_1(x_{2n}, x_{2n-1}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n-2}), d_2(y_{2n-1}, y_{2n-2})\} \tag{3.8}$$

If follows from inequalities (3.5), (3.6), (3.7) and (3.8) that

$$d_1(x_{2n+1}, x_n) \leqslant c \max\{d_1(x_n, x_{n-1}), d_2(y_n, y_{n-1})\}$$

$$d_2(y_{n+1}, y_n) \leqslant c \max\{d_1(x_n, x_{n-1}), d_2(y_n, y_{n-1})\}$$

and an easy induction argument, shows that,

$$d_1(x_{n+1}, x_n) \leqslant c^{n-1} \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

$$d_2(y_{n+1}, y_n) \leqslant c^{n-1} \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

For $n = 1, 2, \ldots$ since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits $z \in X$ and $w \in Y$. Now suppose that A is continuous, then,

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w$$

and so $Az = w$ using inequality (3.1), we have,

$$\begin{aligned} d_1(Sw, x_{2n})\, d_1(SAz, x_{2n+1}) &= d_1(Sw, Ty_{2n})\, d_1(SAz, TBx_{2n}) \\ &\leqslant c \max\{d_1(Sw, Ty_{2n})\, d_2(Az, Bx_{2n}), \\ &\quad d_1(x_2n, Sw)\, d_2(y_{2n}, Az), \\ &\quad d_1(z, x_{2n})\, d_1(Sw, Ty_{2n}), \\ &\quad d_1(Sw, SAz)\, d_1(Ty_{2n}, TBx_{2n})\} \\ &\leqslant c \max\{d_1(Sw, x_{2n})\, d_2(Az, y_{2n+1}), \\ &\quad d_1(x_{2n}, Sw)\, d_2(y_{2n}, Az), \\ &\quad d_1(z, x_{2n})\, d_2(Sw, x_{2n}), \\ &\quad d_1(Sw, SAz)\, d_2(x_{2n}, x_{2n+1})\} \end{aligned}$$

Letting $n \to \infty$, we have

$$d_1(Sw, z)\, d_1(SAz, z) \leqslant 0$$

Thus, $\quad d(Sw, z) = 0 \quad$ or $\quad d(SAz, z) = 0$

So, $\quad Sw = z \quad$ or $\quad SAz = z \implies Sw = z$

Applying inequality (3.2), we have,

$$\begin{aligned} d_2(Az, y_{2n+1})\, d_2(BSw, y_{2n+1}) &= d_2(Az, Bx_{2n})\, d_2(BSw, ATy_{2n}) \\ &\leqslant c \max\{d_1(Sw, Ty_{2n})\, d_2(Az, Bx_{2n}), \\ &\quad d_1(x_{2n}, Sw)\, d_2(y_{2n}, Az), \\ &\quad d_2(w, y_{2n})\, d_2(Az, Bx_{2n}), \\ &\quad d_2(Az, BSw)\, d_2(Bx_{2n}, ATy_{2n})\} \\ &\leqslant c \max\{d_1(Sw, x_{2n})\, d_2(Az, y_{2n+1}), \\ &\quad d_1(x_{2n}, Sw)\, d_2(y_{2n}, Az), \\ &\quad d_2(w, y_{2n})\, d_2(Az, y_{2n+1}), \\ &\quad d_2(Az, BSw)\, d_2(y_{2n+1}, y_{2n+1})\} \end{aligned}$$

Letting $n \to \infty$, we have

$$d_2(Az, w)\, d_2(BSw, w) \leqslant 0$$

Thus, either $d_2(Az, w) = 0$ or $d_2(BSw, w) = 0$
So $Az = w$ $BSw = w$ or $Bz = w$

Again using inequality (3.1), we have

$$\begin{aligned}[d_1(z, Tw)]^2 &= d_1(Sw, Tw)\, d_1(SAz, TBz) \\ &\leqslant c \max\{d_1(Sw, Tw)\, d_2(Az, Bz), \\ &\quad d_1(z, Sw)\, d_2(w, Az), d_1(w, w)\, d_1(Sw, Tw), \\ &\quad d_1(Sw, SAz)\, d_1(Tw, TBw)\}\end{aligned}$$

Letting $n \to \infty$, we have

$$[d_1(z, Tw)]^2 \leqslant 0$$

So $$d_1(z, Tw) = 0$$

Thus $Z = Tw$ or $Tw = z$. The same results ofcourse hold if one of the mapping B, S, T is continuous instead of A.

3.1. **Uniqueness.** Suppose that TB has a second fixed point z'. Then inequality (3.1) and (3.2), we have,

$$\begin{aligned}[d_1(z, z')]^2 &= d_1(z, z')\, d_1(SAz, TBz') \\ &\leqslant c \max\{d_1(z, z')\, d_2(Az, Bz'), d_1(z', z)\, d_2(Bz', Az), \\ &\quad d_1(z, z')\, d_1(z, z'), d_1(z, SAz)\, d_1(z', TBz')\} \\ [d_1(z, z')]^2 &\leqslant c \max\{d_1(z, z')\, d_2(Az, Bz'), d_1(z, z')\, d_1(z, z')\}\end{aligned}$$

$$\Longrightarrow \qquad d_1(z, z') \leqslant c d_2(Az, Bz') \tag{3.9}$$

Further applying inequality (3.2), we have,

$$\begin{aligned}[d_2(Az, Bz')]^2 &= d_2(Az, Bz')\, d_2(BSBz, ATAz') \\ &\leqslant c \max\left\{d_1(z, z')\, d_2(Az, Bz'), [d_2(Az, Bz')]^2\right\}\end{aligned}$$

$$\Longrightarrow \qquad d_2(Az, Bz') \leqslant c d_1(z, z') \tag{3.10}$$

It now follows from inequalities (3.9) and (3.10) that

$$d_1(z, z') \leqslant c d_2(Az, Bz') \leqslant c^2 d_1(z, z')$$

and so $z = z'$ since $c < 1$, proving the uniqueness of the fixed point z of TS. It follows similarly that z is the unique fixed point of SA and w is the unique fixed point of BS and AT. This complete the proof of the theorem. □

Corollary 3.1. *Let (X, d) be a complete metric space and let A, B be a continuous mapping of X into X satisfying inequality*

$$d_1(Ay, By')\, d_1(A^2x, B^2x') \leqslant c \max\{d_1(Ay, By')\, d_1(Ax, Bx'),\ d_1(x', Ay)\, d_1(y', Ax),\ d_1(x, x')\, d_1(Ay, By'),\ d_1(Ay, A^2x)\, d_1(By', B^2y')\}$$

$\forall\ x, x', y, y' \in X$, where $0 \leqslant c < 1$. Then T has a unique fixed point $u \in X$.

Proof. It follows from the theorem with $(X, d_1) = (Y, d_2)$ and $S = A, T = B$, then A^2 and B^2 have unique fixed points u. Then $A^2(Au) = A(A^2u)$ and we see that Au is also a fixed point of A^2. Since fixed point is unique. We must have $Au = u$. Similarly we can prove for B. □

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