A new class of Weighted Generalized Beta distribution of first kind and its structural properties

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In this research paper, a new class of weighted Generalized Beta distribution of first kind (WGBD1) is introduced. A Size biased Generalized Beta distribution of first kind and area biased Generalized Beta distribution of first kind are the particular cases of the weighted Generalized Beta distribution of first kind, taking the weights as the variate values has been defined. A new distribution which contains as a special case is introduced .The characterizing properties of the model are derived and obtained. The estimates of the parameters of Weighted Generalized Beta distribution of first kind (WGBD1) are obtained by employing a new method of moments. Also, a test for detecting the size-baisedness of this new model is conducted.

KEYWORDS: Generalized Beta distribution of first kind, Beta function, weighted generalized beta distribution, moment estimator, Likelihood ratio test.

ABSTRACT

1. INTRODUCTION

The probability density function (pdf) of the generalized beta distribution of first kind (GBD1) is given by:

$$f(x;a,b,p,q)=\frac{a}{b^{ap}\beta(p,.q)}x^{ap-1}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1} \qquad for\ x>0$$
$$= \ 0, otherwise \qquad (1.1)$$

where a,p,q are shape parameters and b is a scale parameter, $\beta(p,q)=\frac{\Gamma p\Gamma q}{\Gamma p+q}$ is a beta function, a,b,p,q are positive real values.

The cth moment of generalized beta distribution of first kind is given by J.B.McDonald (1995):

$$E(X^c) = \frac{b^c \beta\left(p + \frac{c}{a}, q\right)}{\beta(p,q)} \qquad (1.2)$$

Put c =1 in relation (1. 2), we have

$$E(X) = \frac{b\beta\left(p + \frac{1}{a}, q\right)}{\beta(p,q)} \qquad (1.3)$$

Beta distributions are very versatile and a variety of uncertainties can be usefully modelled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modelled by finite range distributions, see for example Barlow and Proschan (1975). Many generalizations of beta distributions involving algebraic and exponential functions have been proposed in the literature; see in Johnson et al. (1995) and Gupta and NadarSajah (2004) for detailed accounts. The generalized beta distribution of first kind J.B.McDonald (1984) is a very flexible four parameter distribution. It captures the characteristics of income distribution including skewness, peakedness in low-middle range, and long right hand tail. The Generalized Beta distribution of first kind includes several other distributions as special or limiting cases, such as generalized gamma (GGD), Dagum, beta of the second kind (BD2), Sing-Maddala (SM), gamma, Weibull and exponential distributions.

2. Derivation of Weighted Generalized Beta Distribution of first kind

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to measure of the unit size, resulting distribution is called size-biased. Size biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher (1934) to model ascertainment bias, these were later formalized in a unifying theory by Rao (1965). These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non –experimental, non –replicated and non –random categories.

If the random variable X has distribution $f(x;\theta)$, with unknown parameter θ, then the corresponding weighted distribution is of the form

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu'_c} \qquad (2.1)$$

$$\mu'_c = \int x^c f(x;\theta)dx \qquad \text{For continuous series} \qquad (2.2)$$

$$\mu_c' = \sum_{i=1}^{n} x^c f(x;\theta)dx \qquad \text{For discrete series.}$$

When c = 1 and 2, we get the size –biased and area biased - distributions respectively.

A Weighted generalized Beta distribution of first kind (WGBD1) is obtained by applying the weights x^c , to the weighted Generalized Beta distribution of first kind.

We have from relation (2.1), we have

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu_c'}$$

$$f_w^*(x;a,b,p,q) = \int_0^\infty x^c \frac{a \quad x^{ap-1}}{b^{ap}\,\beta(p,q)}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} . \frac{\beta(p,q)}{b^c\,\beta\left(p+\frac{c}{a},q\right)}\ dx$$

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{ap+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

Where $f_w^*(x;a,b,p,q)$ represents a probability density function. This gives the weighted generalized beta distribution of first kind (WGBD1) as:

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{ap+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} \tag{2.3}$$

Where a,p,q are shape parameters and b is a scale parameter, $\beta(p+\frac{c}{a},q) = \frac{\Gamma\left(p+\frac{c}{a}\right)\Gamma q}{\Gamma\left(p+q+\frac{c}{a}\right)}$ is a beta function, a,b,p,q are positive real values.

Special cases:

1. The distribution like the weighted beta distributions of first kind as special case when $a=b=1$, then the probability density function is given as:

$$f_w^*(x;p,q) = \frac{1}{\beta(p+c,q)} x^{p+c-1}\,(1-x)^{q-1},\ p>0, q>0 \tag{2.4}$$

2. The distribution like the Size-biased beta distribution of first kind as particular case when $a=b=c=1$, then the probability density function is given as:

$$f_s^*(x;p,q)= \frac{1}{\beta(p+1,q)} x^p (1-x)^{q-1}, p>0, q>0 \tag{2.5}$$

3. The distribution like the area-biased beta distribution of first kind as particular case when $a=b=1, c=2$, then the probability density function is given as:

$$f_A^*(x;p,q)= \frac{1}{\beta(p+2,q)} x^{p-1} (1-x)^{q-1}, p>0, q>0 \tag{2.6}$$

$\beta(p+2,q)=\dfrac{\Gamma p+2\Gamma q}{\Gamma p+q+2}$ is a beta function, a,b,p,q are positive real values.

3. Structural properties of weighted Generalized beta distribution of first kind:

The rth moment of weighted generalized beta distribution of first kind (2.3) about origin is obtained as:

$$\mu_r' = \int_0^\infty x^r f_w(x;a,b,p,q)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{a\,x^{ap+c-1}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \int_0^\infty \frac{a\,x^{ap+r+c-1}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty \left(\frac{x}{b}\right)^{ap+r+c-1}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty \left[\left(\frac{x}{b}\right)^a\right]^{p+\frac{r+c-1}{a}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

Put $\left(\frac{x}{b}\right)^a = t,\ then\ \ x=bt^{\frac{1}{a}}, dx=\frac{b}{a}t^{\frac{1}{a}-1}dt$

$$\mu_r' = \frac{b^r}{\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty [t]^{p+\frac{r+c}{a}-1}[1-t]^{q-1}dt$$

$$\mu_r' = \frac{b^r}{\beta\left(p+\frac{c}{a},q\right)}\beta\left(p+\frac{r+c}{a},q\right) \qquad (3.1)$$

Using the equation (3.1), the mean and second moment of the WGBD1 is given by

$$\mu_1' = \frac{b\,\beta\left(p+\frac{c+1}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} \qquad (3.2)$$

$$\mu_2' = \frac{b^2\,\beta\left(p+\frac{c+2}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} \qquad (3.3)$$

Using the equation (3.2) and (3.3), the variance of the WGBD1 is given by

$$\mu_2 = b^2\left[\frac{\beta\left(p+\frac{c+2}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} - \left[\frac{\beta\left(p+\frac{c+1}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)}\right]^2\right] \qquad (3.4)$$

The Coefficient of variation of Weighted Generalized Beta Distribution of first kind.

$$CV = \sqrt{\frac{\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)}{\beta^2\left(p+\frac{c+1}{a},q\right)}-1} \qquad (3.5)$$

The Coefficient of skewness and Kurtosis of weighted Generalized Beta Distribution of first kind are obtained by using these formulas:

$$CS = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'}{\sigma^3} \qquad (3.6)$$

$$CS = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3' - 4\mu\mu_3' + 6\mu_1'^2\mu_2' - 3\mu^4}{\sigma^3} \tag{3.7}$$

The mode of weighted generalized beta distribution of first kind is given as:

The probability distribution of weighted Generalized Beta distribution of first kind is:

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{ap+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

In order to discuss monotonicity of weighted generalized beta distribution of first kind. We take the logarithm of its pdf:

$$\ln(f_w(x:a,b,p,q)) = \ln\left(\frac{a}{b^{ap+c}\beta\left(p+\frac{c}{a},q\right)}\right) + \ln x^{ap+c-1} + \ln\left\{\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}\right\} \tag{3.8}$$

Where C is a constant. Note that

$$\frac{\partial \ln f_w^*(x;a,b,p,q)}{\partial x} = \frac{(ap+c-1)\left(b^a-x^a\right)-(q-1)ax^a}{x\left(b^a-x^a\right)}$$

Where a,p,q are shape parameters and b is a scale parameter, It follows that

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} > 0 \Leftrightarrow x < \frac{(ap+c-1)^{\frac{1}{a}}\left(b^a-x^a\right)^{\frac{1}{a}}}{[a(q-1)]^{\frac{1}{a}}}$$

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} < 0 \Leftrightarrow x > \frac{(ap+c-1)^{\frac{1}{a}}\left(b^a-x^a\right)^{\frac{1}{a}}}{[a(q-1)]^{\frac{1}{a}}}$$

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} = 0 \Leftrightarrow x = \frac{(ap+c-1)^{\frac{1}{a}}\left(b^a-x^a\right)^{\frac{1}{a}}}{[a(q-1)]^{\frac{1}{a}}}$$

The mode of weighted generalized beta distribution of first kind is:

$$x = \frac{(ap+c-1)^{\frac{1}{a}}\left(b^a-x^a\right)^{\frac{1}{a}}}{[a(q-1)]^{\frac{1}{a}}} \tag{3.9}$$

The harmonic mean of weighted generalized beta distribution of first kind is given as:

The probability distribution of weighted Generalized Beta distribution of first kind is:

$$f_w^{*}(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{a+c-1}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1}$$

The harmonic mean (H) is given as:

$$\frac{1}{H} = \int_0^\infty \frac{1}{x}\ f_w(x;a,b,p,q)dx$$

$$\frac{1}{H} = \int_0^\infty \frac{1}{x}\frac{a\,x^{ap+c-1}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}dx$$

$$\frac{1}{H} = \int_0^\infty \frac{a\,x^{ap+c-2}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}dx$$

$$\frac{1}{H} = \frac{a}{b^2\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty \left[\left(\frac{x}{b}\right)^{a}\right]^{p+\frac{c-2}{a}}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}dx$$

Put $\left(\frac{x}{b}\right)^{a} = t,\ then\ \ x = bt^{\frac{1}{a}}, dx = \frac{b}{a}t^{\frac{1}{a}-1}dt$

$$\frac{1}{H} = \frac{a}{b\ \beta\left(p+\frac{c}{a},q\right)}\int_0^\infty [t]^{p+\frac{c-1}{a}-1}[1-t]^{q-1}dt$$

$$\frac{1}{H} = \frac{\beta\left(p+\frac{c-1}{a},q\right)}{b\beta\left(p+\frac{c}{a},q\right)}$$

$$H = \frac{b\beta\left(p+\frac{c}{a},q\right)}{\beta(p+\frac{c-1}{a},q)} \qquad (3.10)$$

4. Estimation of parameters of the weighted Generalized Beta Distribution of first kind.

In this section, we obtain estimates of the parameters for the weighted Generalized Beta distribution of first kind by employing the new method of moment (MOM) estimator.

4.1 New Method of Moment Estimators

Let X_1, X_2, X_3… Xn be an independent sample from the WGBD1. The method of moment estimators are obtained by setting the row moments equal to the sample moments, that is $E(X^r) = M_r$ where is the sample moment M_r corresponding to the $E(X^r)$ The following equations are obtained using the first and second sample moments.

$$\frac{1}{n}\sum_{j=1}^{n} X_j = \frac{b\,\beta\left(p+\frac{c+1}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} \tag{4.1}$$

$$\frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} = \frac{b^2\,\beta\left(p+\frac{c+2}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} \tag{4.2}$$

Case 1.When p and q are fixed and a=1, then

$$\frac{\overline{X}}{M_2} = \frac{\Gamma(p+q+c+1)}{b\Gamma(p+c+1)}$$

$$\hat{b} = \frac{M_2}{\overline{X}}\left[1+\frac{q}{p+c+1}\right] \tag{4.3}$$

Case 2. When p and b are fixed and a=1, then dividing equation (17) by (18), we have:

$$\frac{\overline{X}}{M_2} = \frac{\Gamma(p+q+c+1)}{b\Gamma(p+c+1)}$$

$$\hat{q} = (p+c+1)\left[\frac{b\overline{X}}{M_2}-1\right] \tag{4.4}$$

Case 3: When b and q are fixed and a=1, then dividing equation (17) by (18), we have:

$$\frac{\overline{X}}{M_2} = \frac{\Gamma(p+q+c+1)}{b\Gamma(p+c+1)}$$

$$\hat{p} = \frac{qM_2}{b\,\overline{X} - M_2} - (c+1) \tag{4.5}$$

Case 4. When p and q are fixed, b=1 then we can calculate the value of $\hat{a}$ estimator by numerical methods.

5. Test for weighted generalized beta distribution of second kind.

Let X_1, X_2, X_3… Xn be a random samples can be drawn from generalized beta distribution of first kind or weighted generalized beta distribution of first kind. We test the hypothesis $H_o : f(x) = f(x,a,b,p,q)\, againest H_1 : f(x) = f_w^{\,*}(a,b,p,q)$.

To test whether the random sample of size n comes from the generalized beta distribution of first kind or weighted generalized beta distribution of first kind the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f^*{}_w(x;a,b,p,q)}{f(a,b,p,q)}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\dfrac{a x^{ap+c-1}}{b^{ap+c}\beta\left(p+\frac{c}{a},q\right)}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}}{\dfrac{a x^{ap-1}}{b^{ap}\beta(p,q)}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\beta(p,q)}{b^c\beta\left(p+\frac{c}{a},q\right)}.x^c$$

$$\Delta = \left[\frac{\beta(p,q)}{b^c\beta\left(p+\frac{c}{a},q\right)}\right]^n \prod_{i=1}^{n} x_i^{\,c} \tag{5.1}$$

We reject the null hypothesis.

$$\left[\frac{\beta(p,q)}{b^c\Gamma\left(p+\frac{c}{a},q\right)}\right]^n \prod_{i=1}^{n} x_i^{\,c} \;\; > k$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^{*} = \prod_{i=1}^{n} x_i^{\,c} > k^{*}, where k^{*} = k\left[\frac{b^{c}\beta\left(p+\frac{c}{a},q\right)}{\beta(p,q)}\right]^{n} > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution. Also, we can reject the reject the null hypothesis, when probability value s given by:

$$P(\Delta^{*} > \lambda^{*}), Where\ \lambda^{*} = \prod_{i=1}^{n} x_i$$

Is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the test statistic Δ^{*}.

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