Solving Fuzzy Differential Equations using Ralston's Method with Triangular Fuzzy Number

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ABSTRACT

In this paper, an attempt has been made to determine a numerical solution for the first order fuzzy differential equations by using Ralston's method which evaluates the integrand f(x,y) twice for each step. The accuracy and efficiency of the proposed method is illustrated by solving a fuzzy initial value problem with triangular fuzzy number.

Key words: Fuzzy Differential Equations, Ralston's Second order Method, Triangular Fuzzy Number

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1.INTRODUCTION

Fuzzy differential equations are a natural way to model dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations,which appear in many applications.The concept of fuzzy derivative was first introduced by S.L.Chang and L.A.Zadeh in[6].D.Dubois and Prade [7] discussed differentiation with fuzzy features.M.L.puri and D.A.Ralescu[19] and R.Goetschel and W.Voxman[10] contributed towards the differential of fuzzy functions.The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva[11,12] and by S.Seikkala[20].Recently many research papers are focused on numerical solution of fuzzy initial value problems (FIVPS).Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M.Friedman, A.Kandel [14] through Euler method and by S.Abbasbandy and T.Allahviranloo [1] by Taylor method.Runge –Kutta methods have also been studied by authors [2,17].V.Nirmala,N.Saveetha,S.Chenthurpandiyan discussed on numerical Solution of Fuzzy Differential Equations by Runge-Kutta Method with Higher order derivative approximations[16].Numerical solutions of fuzzy differential equations by Runge-kutta method of order two was discussed by V,Parimala,P.Rajarajeswari,V.Nirmala[18].

This paper is organised as follows:In section 2 some basic results of fuzzy numbers and definitions of fuzzy derivative are given.In section 3 the fuzzy initial value problem is being discussed.Section 4 describes the general structure of the Ralston's method.In section 5, the Ralston's method,in particular,for solving fuzzy initial value problem has been discussed. Finally the applicability of the method is demonstrated by determining the numerical solution of the problem by applying the proposed method.

2.PRELIMINARIES

2.1 Definition:(FUZZY NUMBER)

An arbitrary fuzzy number is represented by an ordered pair of functions ($\underline{u}$(r),$\bar{u}$(r)) for all rϵ[0,1] which satisfy the following conditions.

i)$\underline{u}$(r) is a bounded left continuous non-decreasing function over [0,1] with respect to any r.

ii)$\bar{u}$(r)is a bounded right continuous non-decreasing function over [0,1] with respect to any r.

iii)$\underline{u}$(r) ≤ $\bar{u}$(r) for all rϵ[0,1] then the r-level set is $[u]_r = \{ x \backslash u(x) \geq r \}$; $0 \leq r \leq 1$

Clearly, $[u]_0 = \{ x \backslash u(x) > 0 \}$ *is compact,* which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \bar{u}(r)]$

2.2 Definition: (TRIANGULAR FUZZY NUMBER)

A triangular fuzzy number u is a fuzzy set in E that is characterized by an ordered triple $(u_l,u_c,u_r)\epsilon R^3$ with $u_l \leq u_c \leq u_r$ such that $[u]_0=[u_l;u_r]$ and $[u]_1=\{u_c\}$.

The Membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \frac{x-u_l}{u_c-u_l} & ; u_l \leq x \leq u_c \\ 1 & ; \quad x = u_c \\ \frac{u_r-x}{u_r-u_c} & ; \ u_c \leq x \leq u_r \end{cases}$$

. we have :

(1) $u > 0$ *if* $u_l > 0$;
(2) $u \geq 0$ *if* $u_l \geq 0$;
(3) $u < 0$ *if* $u_c < 0$;and
(4) $u \leq 0$ *if* $u_c \leq 0$.

2.3:Definition: (∝ - Level Set)

Let I be the real interval. A mapping y:I→ E is called a fuzzy process and its ∝ - *level Set* is denoted by $[y(t)]_\propto = [\underline{y}(t;\propto), \bar{y}(t;\propto)]$, t ∈ I, 0<∝< 1

2.4:Definition: (Seikkala Derivative)

The Seikkala derivative $y'(t)$ of a fuzzy process is defined by $[y'(t)]_\propto = [\underline{y}'(t;\propto), \bar{y}'(t;\propto)]$, t ∈ I, 0<∝≤ 1 provided that this equation defines a fuzzy number, as in [19]

2.5:Lemma:

If the sequence of non-negative numbers $\{w_n\}_{n=0}^N$,satisfy

$|Wn+1| \leq A\,|Wn| + B$, $0 \leq n \leq N-1$ for the given positive constants A and B, then

$|Wn| \leq A^n |W_0| + B\,\frac{A^n-1}{A-1}$, $0 \leq n \leq N$

2.6:Lemma:

If the sequence of non-negative numbers $\{w_n\}_{n=0}^N$,$\{V_n\}_{n=0}^N$satisfy

$|Wn+1| \leq |Wn| + A\max\{|W_n|, |V_n|\} + B$

$|V_{n+1}| \leq |V_n| + A\, max\{|W_n|, |V_n|\} + B$
for the given positive constants A and B, then $U_n = |W_n| + |V_n|$, $0 \leq n \leq N$
we have $U_n \leq \overline{A^n} U_0 + \overline{B}\frac{\overline{A^n}-1}{\overline{A}-1}$, $0 \leq n \leq N$ where $\overline{A}=1+2A$ and $\overline{B}=2B$.

2.7:Lemma

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$,
$D(y(t_{n+1}), y^0(t_{n+1})) \leq h^2L(1+2C)$, where L is a bound of partial derivatives of F and G, and
$C = \max\left\{\left|G[t_N, \underline{y}(t_N; r), \overline{y}(t_{N-1}; r)]\right|, r \in [0,1]\right\} < \infty$

2.8:Theorem

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$, the numerical solutions of $\underline{y}(t_{n+1}; r)$ and $\overline{y}(t_{n+1}; r)$ converge to the exact solutions $\underline{Y}(t; r)$ and $\overline{Y}(t; r)$ uniformly in t.

2.9:Theorem

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \leq r \leq 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n; r)$ and $\overline{y}^{(j)}(t_n; r)$ converge to the numerical solutions $\underline{y}(t_n; r)$ and $\overline{y}(t_n; r)$ in $t_0 \leq t_n \leq t_N$, when $j \to \infty$.

3.FUZZY INITIAL VALUE PROBLEM

Consider a first-order fuzzy initial value problem is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \qquad (3.1)$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of $y(t)$ for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)],$$
$$[y(t_0)]_r = [\underline{y}(t_0; r), \overline{y}(t_0; r)],\ r \in (0,1]$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$\underline{f}(t, y) = F[t, \underline{y}, \overline{y}]$,

$\overline{f}(t, y) = G[t, \underline{y}, \overline{y}]$.

Because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \overline{y}(t; r)] \qquad (3.2)$$

$$\overline{f}(t, y(t); r) = G[t, \underline{y}(t; r), \overline{y}(t; r)] \qquad (3.3)$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = sup\{y(t)(\tau) \backslash s = f(t, \tau)\},\ s \in R \qquad (3.4)$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)],\ r \in (0,1], \qquad (3.5)$$

where

$\underline{f}$ *(t, y(t); r) = min { f(t, u)|u* ∈$[y(t)]_r$} (3.6)

$\overline{f}$ *(t, y(t); r) = max { f(t, u)|u* ∈$[y(t)]_r$}. (3.7)

3.1:Definition

A function f: $R \rightarrow R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, δ > 0 such that |t – to| <δ ⇒D [f(t), f(t_0)] < ∈ exists.

Throughout this paper we also consider fuzzy functions which are continuous in metric *D*. Then the continuity of *f(t,y(t);r)* guarantees the existence of the definition of *f(t,y(t);r)* for $t \in [t_o, T]$ and $r \in [0,1]$ [10]. Therefore, the functions *G* and *F* can be definite too.

4.RALSTON'S SECOND ORDER METHOD

Ralston's second order method is a Runge-kutta method for approximating the solution of the initial value problem $y'(t) = f(t, y(t)), y(t_0) = y_0$ which evaluates the integrand f(x,y) twice for each step. The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as $y_{n+1} - y_n = \sum_{i=o}^{m} w_i k_i$ (4.1)

Where w_i's are constant for all i and k_i= hf(t_n+$a_i h$,y_n+$\sum_{j=1}^{i-1} c_{ij} k_j$) (4.2)

Most efforts to increase the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required[5].The method proposed by Goeken.D and Johnson.O[9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta $k_i$terms(i > 0) to obtain a higher order of accuracy without a corresponding increase in evaluations of 'f ', but with the addition of evaluations of f'.

The Ralston's second order method for step n+1 is given by considering

Consider y(t_{n+1})=y(t_n)+$W_1 k_1$+$W_2 k_2$ (4.3)

where $k_1 = h f(t_n, y(t_n))$ (4.4)

$k_2 = h f(t_n + c_2 h, y(t_n) + a_{21} k_1)$ (4.5)

and the parameters W_1,W_2,a_{21} are chosen to make y_{n+1} closer to y(t_{n+1}). Using Taylor's series expansion we calculate the value of parameters as $a_{21} = \frac{2}{3}$,$c_2 = a_{21} = \frac{2}{3}$ and the values of the other parameters $W_1 = \frac{1}{4}$ & $W_2 = \frac{3}{4}$.

The Ralston's Second order method is given by

$y(t_{n+1}) = y(t_n) + \frac{k_1 + 3k_2}{4}$ (4.6)

where $k_1 = h f(t_n, y(t_n))$ (4.7)

$k_2 = h f(t_n + \frac{2}{3}h, y(t_n) + \frac{2}{3} k_1)$ (4.8)

5. RALSTON'S SECOND ORDER METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

Let the exact solution $[Y(t)]_r$ $[\underline{Y}(t; r), \overline{Y}(t; r)]$,is approximated by some $[y(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)]$. The grid points at which the solutions is calculated are h=$\frac{T-t_0}{N}$, $t_i = t_0 + ih$; $0 \leq i \leq N$

From 4.6 to 4.8 we define

$\underline{y}(t_{n+1}, r)$-$\underline{y}(t_n; r)$= $\frac{\underline{k_1}(t_n, y(t_n, r)) + 3\underline{k_2}(t_n, y(t_n, r))}{4}$

where

$k_1 = h\, F\,(t_n\,,\, \underline{y}(t_n; r), \overline{y}\,(t_n; r))$

$k_2 = h\, F(t_n + \frac{2h}{3}, \underline{y}(t_n;\, r) + \frac{2}{3}k_1, \overline{y}(t_n;\, r) + \frac{2}{3}k_1)$ (5.1)

and

$\overline{y}(t_{n+1}, r)$-$\overline{y}(t_n; r)$)= $\frac{\overline{k_1}(t_n, y(t_n, r)) + 3\overline{k_2}(t_n, y(t_n, r))}{4}$

where

$k_1 = h\, G\,(t_n\,,\, \underline{y}(t_n; r), \overline{y}\,(t_n; r))$

$k_2 = h\, G(t_n + \frac{2h}{3}, \underline{y}(t_n;\, r) + \frac{2}{3}k_1, \overline{y}(t_n;\, r) + \frac{2}{3}k_1)$ (5.2)

we define

F[t, y(t, r)]= $\frac{\underline{k_1}(t, y(t, r)) + 3\underline{k_2}(t, y(t, r))}{4}$ (5.3)

G[t, y(t, r)]= $\frac{\overline{k_1}(t, y(t, r)) + 3\overline{k_2}(t, y(t, r))}{4}$ (5.4)

Therefore we have

$\underline{Y}(t_{n+1}, r) = \underline{Y}(t_n; r) + F[t_n, Y(t_n; r)]$ (5.5)

$\overline{Y}(t_{n+1}, r) = \overline{Y}(t_n; r) + G[t_n, Y(t_n; r)]$

and

$\underline{y}(t_{n+1}, r) = \underline{y}(t_n; r) + F[t_n, y(t_n; r)]$ (5.6)

$\overline{y}(t_{n+1}, r) = \overline{y}(t_n; r) + G[t_n, y(t_n; r)]$

Clearly $\underline{y}(t; r)$and $\overline{y}(t; r)$converge to $\underline{Y}(t; r)$ and $\overline{Y}(t; r)$ whenever h→0

6.NUMERICAL EXAMPLE

Consider fuzzy initial value problem

$$\begin{cases} y'(t) = y(t), \quad t \geq 0 \\ y(0) = (0.75 + 0.25r\,, 1.125 - 0.125r) \end{cases} \quad (6.1)$$

The exact solution is given by

Y(t;r) = [$(0.75 + 0.25r)e^t$, (1.125-0.125r)e^t]

At t=1we get

Y(1;r) = [$(0.75 + 0.25r)e$, (1.125-0.125r)e] , 0 ≤ r ≤ 1

The values of exact and approximate solution with h= 0.1 is given in Table : 1. The approximate solutions obtained by the proposed method is plotted in Fig:1

Table:1

r	Exact Solution t=1		Approximation of Ralston's Method using Triangular Fuzzy number (h=0.1)	
	$\underline{Y}$(t; r)	$\overline{Y}$ (t; r)	$\underline{y}$(t; r),	$\overline{y}$ (t; r)
0	2.0387113	, 3.0580670	2.031184518 ,	3.053340953
0.2	2.1746254	, 2.9901100	2.171264695 ,	2.985488373

0.4	2.3105395 , 2.9221529	2.306968718 , 2.917636911
0.6	2.4464536 , 2.8541959	2.442672761 , 2.849784887
0.8	2.5823677 , 2.7862388	2.578376803 , 2.781932867
1	2.7182818 , 2.7182818	2.71422174 , 2.71422174

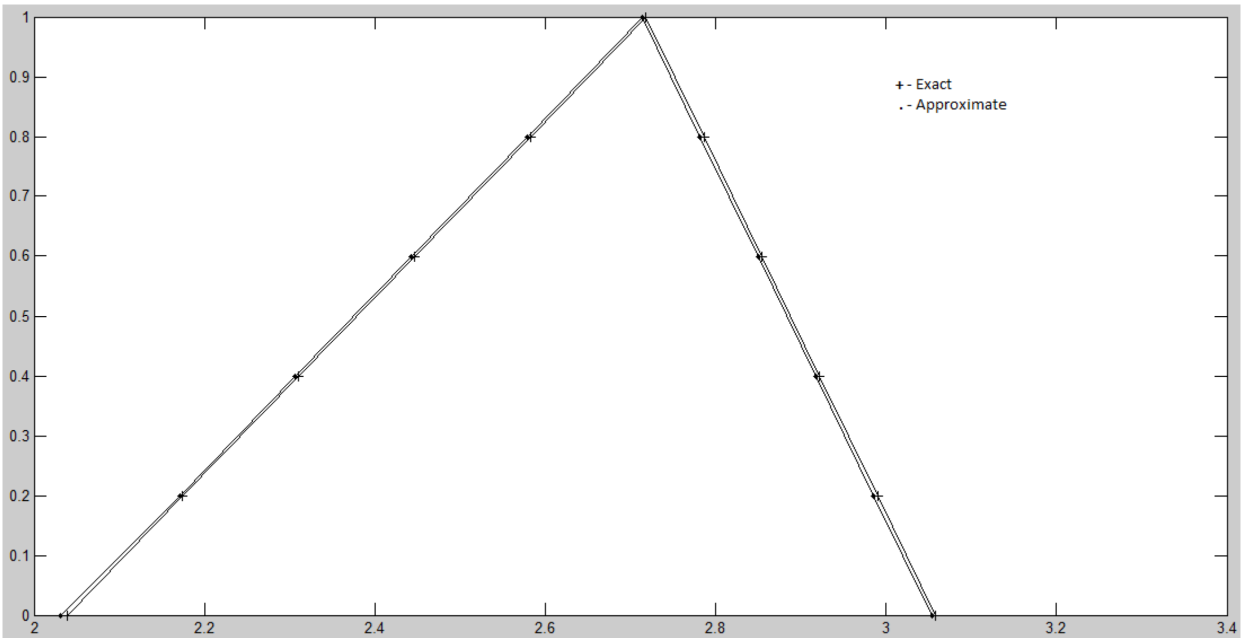

Fig - 1

6.CONCLUSION

In this paper the Ralston's Second order method has being applied for finding the numerical solution of first order fuzzy differential equations using triangular fuzzy number.The efficiency and the accuracy of the Ralston's method have been illustrated by a suitable example. From the numerical example it has been observed that the discreet solutions by the proposed method coincide with the exact solutions.

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