The Modified Power series Method for Solution of Weakly Singular Volterra Integral Equations

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ABSTRACT

In this paper, the modified power series method (MPSM) is presented to solve non-linear weakly singular Volterra integral equations. In the MPSM the solution is obtained in the series form that converges to the exact solution if such a closed form solution exists. The MPSM is simple to understand and easy to implement using computer packages and does not require any restrictive assumptions for nonlinear terms. The main contribution of the current paper is to obtain the exact solution rather than numerical solution as done by some existing techniques. The result showed that the method is overcomes the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM). It does not require to calculate Lagrange multiplier as in Variational Iteration Method (VIM) and no needs to construct a homotopy and solve the corresponding algebraic equations as in Homotopy Perturbation Method (HPM). The software used for the calculations in this study was MATHEMATICA® 9.0.

Key words: Volterra integral equation; Weakly singular integral equation; Exact solution.

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1- INTRODUCTION

Consider the weakly singular Volterra integral equation (WSVIE) of the second kind given by [1].

$$u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)dt \qquad x \in [0, X], \qquad \dots (1)$$

Where $\mu > 0$ and $f(x)$ is a given function, $u(x)$ is the unknown function to be determined.

In this work we will study the second type of weakly singular Volterra integral equation (WSVIE) where the kernel becomes infinite at one or more points of singularities at the range of definition. The (WSVIE) arise in many in the other sciences, such as mathematical physics and chemistry applications such as stereology [2], heat conduction, crystal growth and the radiation of heat from a semi-infinite solid [3]. Eq. (1) has been studied by several works for the case when $\mu > 1$. Certain classes of product integration methods based on Newton–Cotes rules [4]; Diogo et al. [5] considered a fourth-order Hermite-type collocation method and Lima and Diogo [6] developed an extrapolation algorithm, based on Euler's method, Hermite type collocation method [7], Adomian decomposition method (ADM) [8], spline collocation and iterated collocation methods [9,10], the variational iterative method (VIM) [11]. On the other hand, researchers have turned their attention to solving Eq. (1) and most of the methods faced difficulty in solving the equation in the case $0 < \mu \leq 1$, for example Nystrom interpolant method [12], extrapolation methods [13], graded mesh method [14].

There are several methods available in iterative to solve these equations which began to take great interest by researchers in recent years. In this paper, we will study the solution of this set of Eq. (1). First Eq. (1) has unique solution in $C^m[0, X], f \in C^m[0, X]$, and the kernel is a singularity only at $x = 0$, if it was $\mu > 1$ [1]. Secondly Eq. (1) has an infinite set of solutions in C $[0, X]$, which contains only one particular solution belonging to $C^1[0, X]$, and kernel is singular at $x = 0$ and at $t = 0$, for all values of $t > 0$, if it was $0 < \mu \leq 1$[15].

The purpose of this paper is to apply the MPSM to obtain the exact solution of linear and nonlinear weakly singular Volterra integral equation of the second kind, rather than numerical solutions as done by some existing techniques. This method uses simple computations with quite acceptable analytic solutions, which has close agreement with exact solutions.

The outline of the current paper is as follows. In section 2, we review some basic idea of the PSM. In section 3, the modify power series method (MPSM) will be introduced. In section 4, solving weakly singular Volterra integral equation by using the MPSM is presented. In section 5**,** some test examples are solved and finally in section 6 the conclusion is given.

2- BASIC IDEA OF THE PSM

Consider the Volterra integral equation (VIE) of the second kind given by [16, 17].

$$u(x) = f(x) + \int_0^x k(x,t)\,[u(t)]^p\,dt\,, \qquad \ldots(2)$$

The functions $f(x)$ and $k(x,t)$ are known, and $u(x)$ is the unknown function to be determined, also $p > 0$ is a positive integer number. Suppose the solution of Eq. (2) be as.

$$u(x) = c_0 + c_1 x\,, \qquad \ldots(3)$$

Where $c_0 = f(0) = u(0)$ and c_1 is an unknown parameter.

By Substituting Eq. (3) into Eq. (2) with simple calculations, we get

$$(a_1 c_1 - b_1)x + Q(x^2) = 0, \qquad \ldots(4)$$

Where $Q(x^2)$ is a polynomial of order greater than one. By neglecting $Q(x^2)$, we have linear equation of c_1 in the form,

$$a_1 c_1 = b_1\,, \qquad \ldots(5)$$

The parameter c_1 of x in Eq. (3) is then obtained.

In the next step, we assume that the solution of Eq. (2) to be,

$$u(x) = c_0 + c_1 x + c_2 x^2\,, \qquad \ldots(6)$$

Where c_0 and c_1 both are known and c_2 is unknown parameter. By Substituting Eq. (6) into Eq. (2), we obtain:

$$(a_2 c_2 - b_2)x^2 + Q(x^3) = 0, \qquad \ldots(7)$$

Where $Q(x^3)$ is a polynomial of order greater than two. By neglecting $Q(x^3)$, we have linear equation of c_2 in the form,

$$a_2 c_2 = b_2 \,, \qquad \dots (8)$$

The unknown parameter c_2 of x^2 in Eq. (6) is then obtained. Having repeated the above procedure for m iterations, a power series of the following form is derived:

$$u(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m = 0, \qquad \dots (9)$$

Equation (9) is an approximation for the exact solution of the integral equation (2).

It is worth to mention here, we can not apply the PSM for Eq. (1) because we can not achieve $c_0 = f(0)$, when $f(x)$ contain x in denominator, moreover, when the power of x in $f(x)$ is rational number.

3- THE MODIFY POWER SERIES METHOD (MPSM)

In order to overcome the difficulties in PSM, we will be introduced the MPSM.

If $f(x) = a_0 + a_1 x^{e_1} + a_2 x^{e_2} + \cdots,$ ... (10)

Where, $a_i, e_i \in R$, a_i, e_i are known constants, $i = 0,1,2$ ..., since $f(x)$ is given.

Suppose the solution of Eq. (2) be as:

$$u_0(x) = c_0 + c_1 x^{e_1} + c_2 x^{e_2} + \cdots, \qquad \dots (11)$$

Where $c_0, c_1, c_2, \dots$ are unknown parameters, where the c_i have been choosen depend on the terms of $f(x)$ which are known.

By Substituting Eq. (11) into Eq. (2) with simple calculations, we get

$$u_0(x) = f(x) + \int_0^x k(x,t)\,[u_0(t)]^p dt \,, \qquad \dots (12)$$

$$c_0 + c_1 x^{e_1} + c_2 x^{e_2} + \cdots = a_0 + a_1 x^{e_1} + a_2 x^{e_2} + \cdots + \int_0^x k(x,t)\,[c_0 + c_1 t^{e_1} + c_2 t^{e_2} + \cdots]^p dt \,, \qquad \dots (13)$$

After simplifying and equating the like power of x, we achieve the values of c_i. If we achieved the values c_i without remainder then it is exact solution. However, if some terms remain, then the remainder $R_0(x)$ will be

$$R_0(x) = f(x) + \int_0^x k(x,t)\,[u_0(t)]^p dt - u_0(x) \,, \qquad \dots (14)$$

Such that R_0 is the remainder of Eq.(14).

$$R_0(x) = a_0 + a_1 x^{e_1} + a_2 x^{e_2} + \cdots + \int_0^x k(x,t)\,[c_0 + c_1 t^{e_1} + c_2 t^{e_2} + \cdots]^p dt - (c_0 + c_1 x^{e_1} + c_2 x^{e_2} + \cdots), \qquad \dots (15)$$

If $R_0(x) = b_0 + b_1 x^{e_1} + b_2 x^{e_2} + \cdots,$ ... (16)

Where $b_i \in R$, b_i are known constants, $i = 0,1,2$ ...

Suppose the solution of Eq. (2) be as.

$$u_1(x) = d_0 + d_1 x^{e_1} + d_2 x^{e_2} + \cdots, \qquad \dots (17)$$

Where d_i are unknown parameters, which have be chosen depend on the terms of $R_0(x)$ which are known.

By Substituting Eq. (16) and Eq. (17) into Eq. (2) with simple calculations and equating the like power of x, we get

$$u_1(x) = R_0(x) + \int_0^x k(x,t)\,[u_1(t)]^p\,dt\,, \quad \ldots(18)$$

$$d_0 + d_1x^{e_1} + d_2x^{e_2} + \cdots = b_0 + b_1x^{e_1} + b_2x^{e_2} + \cdots + \int_0^x k(x,t)\,[d_0 + d_1t^{e_1} + d_2t^{e_2} + \cdots]^p\,dt\,, \quad \ldots(19)$$

Then we obtain the values of d_i, by repeating the steps in Eqs. (16) – (19) if we obtain another remainders in Eq. (19).
Then the solution of Eq. (2), will be then:

$$u(x) = u_0(x) + u_1(x) + \cdots, \quad \ldots(20)$$

As an application of the MPSM, we will be applied for the following Volterra integral equation.

Example:
Considering the following Volterra integral equation [18]:

$$u(x) = f(x) - \int_0^x (x-t)u(t)dt, \quad \ldots(21)$$

Where, $f(x) = -2 + 3x - x^2$, and the exact solution is $u(x) = -2 + 3\sin(x)$.
By applying the MPSM, we assume the solution of Eq. (21) be as:

$$u_0(x) = c_0 + c_1x + c_2x^2\,, \quad \ldots(22)$$

Substitute Eq. (22) into Eq. (21) we get,

$$c_0 + c_1x + c_2x^2 = -2 + 3x - x^2 - \int_0^x (x-t)(c_0 + c_1t + c_2t^2)dt, \quad \ldots(23)$$

By integrating and solving we get,

$$c_0 + c_1x + c_2x^2 = -2 + 3x - x^2 - \frac{c_0x^2}{2} - \frac{c_1x^3}{6} - \frac{c_2x^4}{12}\,, \quad \ldots(24)$$

After simplifying and equating the like power of x, we achieve the values of c_i, we get,

$c_0 = -2, \quad c_1 = 3\;, c_2 = 0,$

Substitute $c_0 = -2,\ c_1 = 3\ , c_2 = 0,$ in Eq. (22) we get,

$$u_0(x) = -2 + 3x\,, \quad \ldots(25)$$

$$R_0(x) = -2 + 3x - x^2 - \frac{c_0x^2}{2} - \frac{c_1x^3}{6} - \frac{c_2x^4}{12} - (c_0 + c_1x + c_2x^2) = -\frac{x^3}{2}\,, \quad \ldots(26)$$

Suppose the solution of Eq. (21) be as.

$$u_1(x) = c_3x^3\,, \quad \ldots(27)$$

Substitute Eq. (26) and Eq. (27) into Eq. (21),

$$u_1(x) = R_0(x) - \int_0^x (t-x)\big(u_1(t)\big)dt, \quad \ldots(28)$$

By integrating and solving we get,

$$c_3x^3 = -\frac{x^3}{2} - \frac{c_3x^5}{20}\,, \quad \ldots(29)$$

After simplifying and equating the like power of x, we achieve the values of c_3, we get

$$c_3 = -\frac{1}{2}, \qquad \dots(30)$$

Substitute Eq. (30) in Eq. (27) we get,

$$u_1(x) = -\frac{1}{2}x^3, \qquad \dots(31)$$

$$R_1(x) = -\frac{x^3}{2} - \frac{c_3 x^5}{20} - c_3 x^3 \quad = \frac{x^5}{40}, \qquad \dots(32)$$

Suppose the solution of Eq. (21) be as.

$$u_2(x) = c_4 x^5, \qquad \dots(33)$$

Substitute Eq. (32) and Eq. (33) into Eq. (21),

$$u_2(x) = R_1(x) - \int_0^x (x-t)(\ u_2(t)\ )dt, \qquad \dots(34)$$

We get,

$$c_4 x^5 = \frac{x^5}{40} - \int_0^x (x-t)(\ c_4 t^5\ )dt, \qquad \dots(35)$$

After simplifying and equating the like power of x, we achieve the values of c_i, we get

$$c_4 x^5 = \frac{x^5}{40} - \frac{c_4 x^7}{42}, \qquad \dots(36)$$

$$c_4 = \frac{1}{40}, \qquad \dots(37)$$

$$u_2(x) = \frac{x^5}{40}, \qquad \dots(38)$$

$$R_2(x) = \frac{x^5}{40} - \frac{c_4 x^7}{42} - c_4 x^5 = -\frac{x^7}{1680}, \qquad \dots(39)$$

By continuing in this manner, the solution of Eq. (21) will be then:

$$u(x) = u_0 + u_1 + u_2 + \cdots = -2 + 3x - \frac{x^3}{2} + \frac{x^5}{40} + \cdots = -2 + 3\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) = -2 + 3\sin(x), \qquad \dots(40)$$

4- SOLVING WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATION BY USING PROPOSED METHOD

In this section, we will be used the MPSM to obtain the exact solution for a weakly singular Volterra integral equation of second kind. Let us consider a form of WSVIE given in Eq. (1).

$$u(x) = f(x) + \int_0^x \frac{t^{\square-1}}{x^{\square}}\, u(t)dt, \qquad \dots(41)$$

If

$$f(x) = a_0 + a_1 x^{e_1} + a_2 x^{e_2} + \cdots, \qquad \dots(42)$$

Where, $a_i, e_i \in R$, a_i, e_i are known constants, $i = 0,1,2,\dots$

Suppose the solution of Eq. (41) be as.

$$u_0(x) = c_0 + c_1 x^{e_1} + c_2 x^{e_2} + \cdots, \qquad \dots(43)$$

Where $c_0, c_1, c_2, \dots$ are unknown parameters.

Now substitute Eq. (42) and Eq. (43) into Eq. (41).

$$u_0(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u_0(t)\, dt\,, \qquad \dots (44)$$

After compensation $u_0(x), f(x)$ we get,

$$c_0 + c_1x^{e_1} + c_2x^{e_2} + \cdots$$
$$= a_0 + a_1x^{e_1} + a_2x^{e_2} + \cdots + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} (c_0 + c_1t^{e_1} + c_2t^{e_2} + \cdots)dt\,, \qquad \dots (45)$$

By integrating and solving we get

$$c_0 + c_1x^{e_1} + c_2x^{e_2} + \cdots = a_0 + a_1x^{e_1} + a_2x^{e_2} + \cdots + \frac{c_0}{\mu} + \frac{c_1x^{e_1}}{\mu + e_1} + \frac{c_2x^{e_2}}{\mu + e_2} + \cdots, \qquad \dots (46)$$

After simplifying and equating the like power of x, we achieve the values of c_i, we get

$$c_0 = \frac{\mu a_0}{\mu - 1}\,, \qquad c_1 = \frac{(\mu + e_1)a_1}{\mu + e_1 - 1}\,, \qquad c_2 = \frac{(\mu + e_2)a_2}{\mu + e_2 - 1}\,, \dots \qquad \dots (47)$$

Substitute $c_0, c_1, c_2, \dots$ in Eq. (43) we get,

$$u_0(x) = \frac{\mu a_0}{\mu - 1} + \frac{(\mu + e_1)a_1}{\mu + e_1 - 1}x^{e_1} + \frac{(\mu + e_2)a_2}{\mu + e_2 - 1}x^{e_2} + \cdots, \qquad \dots (48)$$

After finding $u_0(x)$, we find the remainder $R_0(x)$.

$$R_0(x) = a_0 + a_1x^{e_1} + a_2x^{e_2} + \cdots + \frac{c_0}{\mu} + \frac{c_1x^{e_1}}{\mu + e_1} + \frac{c_2x^{e_2}}{\mu + e_2} + \cdots - (c_0 + c_1x^{e_1} + c_2x^{e_2} + \cdots) = 0 \qquad \dots (49)$$

We conclude that, if $R_0(x) = 0$, then $u_i(x) = 0, i = 1,2,3, \dots$
Thus, the solution of Eq. (41) is,

$$u(x) = u_0(x) = \frac{\mu a_0}{\mu - 1} + \frac{(\mu + e_1)a_1}{\mu + e_1 - 1}x^{e_1} + \frac{(\mu + e_2)a_2}{\mu + e_2 - 1}x^{e_2} + \cdots, \qquad \dots (50)$$

Remark:

It is interesting to point out here that the remainders will be zeros when we apply MPSM for WSVIES.

5- TEST EXAMPLES

In this section the applications of MPSM for the linear and nonlinear WSVIE will be presented.

5.1 LINEAR WSVIE

Let us solve first some test examples for linear WSVIEs by MPSM for two cases:

Case 1: $\mu > 1$:

The MPSM will be implemented to solve some examples of the WSVIE when $\mu > 1$ which already solved numerically by some existing techniques, (ADM) [8], (VIM) [11], and obtain the exact solutions.

Example 1:

Considering the following WSVIE [19]:

$$u(x) = \frac{44}{54}\, x^{-0.5} + x^{-5.9} \int_0^x t^{4.9}\, u(t)dt, \qquad \dots (51)$$

Where, $\mu = 5.9$, $f(x) = \frac{44}{54} x^{-0.5}$, and the exact solution is $u(x) = x^{-0.5}$.
By applying the MPSM, we assume the solution of Eq. (51) be as:

$$u_0(x) = c_1 x^{-0.5} = \frac{c_1}{\sqrt{x}} , \quad \dots (52)$$

Substitute Eq. (52) in Eq. (51).

$$u_0(x) = f(x) + x^{-5.9} \int_0^x t^{4.9}\, u_0(t)\, dt, \quad \dots (53)$$

We get:

$$\frac{c_1}{\sqrt{x}} = \frac{22}{27\sqrt{x}} + x^{-5.9} \int_0^x t^{4.9}\, \frac{c_1}{\sqrt{t}}\, dt, \quad \dots (54)$$

By integrating and solving we get,

$$\frac{c_1}{\sqrt{x}} = \frac{22}{27\sqrt{x}} + \frac{5c_1}{27\sqrt{x}} , \quad \dots (55)$$

After simplifying and equating the like power of *x*, we get the values of c_1.

$$c_1 = 1 , \quad \dots (56)$$

Substitute Eq. (56) in Eq. (52) we get,

$$u_0(x) = \frac{1}{\sqrt{x}} , \quad \dots (57)$$

Now find the remainder $R_0(x)$.

$$R_0(x) = \frac{22}{27\sqrt{x}} + \frac{5c_1}{27\sqrt{x}} - \frac{c_1}{\sqrt{x}} = 0, \quad \dots (58)$$

Since $R_0(x) = 0$, then $u_i(x) = 0,\ i = 1,2,3, \dots$
Thus, the solution of Eq. (51) is,

$$u(x) = u_0(x) = \frac{1}{\sqrt{x}} = x^{-0.5} , \quad \dots (59)$$

Which the exact solution for Eq. (51) [9,19].

Example 2:

Let us consider the following WSVIE given in [19]:

$$u(x) = \frac{65}{75}\, x^3 + \frac{7}{8} x^{3.5} + x^{-4.5} \int_0^x t^{3.5}\, u(t) dt, \quad \dots (60)$$

Where, $\mu = 4.5$, and the exact solution is $u(x) = x^{3.5} + x^3$.
Using the MPSM, we assume the solution of Eq. (60) be as:

$$u_0(x) = c_1 x^3 + c_2 x^{3.5} , \quad \dots (61)$$

Substitute Eq. (61) in Eq. (60) we have,

$$c_1 x^3 + c_2 x^{3.5} = \frac{65}{75}\, x^3 + \frac{7}{8} x^{3.5} + x^{-4.5} \int_0^x t^{3.5}\, (c_1 t^3 + c_2 t^{3.5})\, dt, \quad \dots (62)$$

By integrating and solving we get,

$$c_1 x^3 + c_2 x^{3.5} = \frac{13x^3}{15} + \frac{2c_1 x^3}{15} + \frac{7x^{7/2}}{8} + \frac{c_2}{8} x^{7/2} , \quad \dots (63)$$

After simplifying and equating the like power of *x*, we get the values of $c_1 . c_2$

$c_1 = c_2 = 1$, ... (64)

Substitute Eq. (64) in Eq. (61) we get,

$u_0(x) = x^{3.5} + x^3$, ... (65)

Then, the solution of Eq. (60) is,

$u(x) = u_0(x) = x^{3.5} + x^3$, ... (66)

Which the exact solution for Eq. (60) [9, 19],

Example 3:

Consider the following linear WSVIE given in [19]:

$$u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}}\, u(t)dt, \qquad \text{... (67)}$$

Where $f(x) = \frac{55}{65}\, x^5 + \frac{7}{8} x^{6.5}, \mu = 1.5$, and $u(x) = x^3$ is the exact solution.

Suppose the solution of Eq. (67) be as:

$u_0(x) = c_1\, x^5 + c_2 x^{\frac{13}{2}}$, ... (68)

Substitute Eq. (68) into Eq. (67) we have,

$$c_1\, x^5 + c_2 x^{\frac{13}{2}} = \frac{11}{13}\, x^5 + \frac{7}{8} x^{\frac{13}{2}} + x^{\frac{3}{2}} \int_0^x t^{\frac{1}{2}} (c_1\, t^5 + c_2 t^{\frac{13}{2}})\; dt\,, \qquad \text{... (69)}$$

After simplifying, we get,

$$c_1\, x^5 + c_2 x^{\frac{13}{2}} \;\; = \frac{11}{13}\, x^5 + \frac{7}{8} x^{\frac{13}{2}} \; + \frac{2c_1}{13}\, x^5 + \frac{c_2}{8} x^{\frac{13}{2}}\,, \qquad \text{... (70)}$$

By equating the like power of *x*, we get,

$c_1 = c_2 = 1$, ... (71)

Substitute Eq. (71) in Eq. (68) we get,

$u_0(x) = x^5 + x^{6.5}$, ... (72)

Hence,

$u(x) = u_0(x) = x^5 + x^{6.5}$, ... (73)

Which the exact solution for Eq. (67) [9, 19],

Case 2: $0 < \mu \leq 1$:

In this case the MPSM will be applied to solve some examples of the WSVIE when $0 < \mu \leq 1$ which already solved numerically by some existing techniques and obtain the exact solution.

Example 4:

Considering the WSVIE given in [19]:

$$u\,(x) = x^{1.5} + x + 1 + x^{-0.5} \int_0^x t^{-0.5}\, u(t)dt \;\; , \qquad \text{... (74)}$$

Where $f(x) = x^{1.5} + x + 1, \mu = 0.5$, and $u(x) = 2x^{1.5} + 3x - 1.$ is the exact solution.

By applying the MPSM, we assume the solution of Eq. (74) be as:

$u_0(x) = c_0 + c_1 \boldsymbol{x} + c_2 x^{\frac{3}{2}}$, ... (75)

Substitute Eq. (75) into Eq. (74) we have,

$$c_0 + c_1\boldsymbol{x} + c_2x^{\frac{3}{2}} = x^{1.5} + x + 1 + x^{\frac{-1}{2}}\int_0^x t^{\frac{-1}{2}}(c_0 + c_1t + c_2t^{\frac{3}{2}})\ dt\ , \quad \dots(76)$$

After simplifying, we get,

$$c_0 + c_1x + c_2x^{\frac{3}{2}} = x^{\frac{3}{2}} + x + 1 + 2c_0 + \frac{2}{3}c_1x + \frac{1}{2}c_2x^{\frac{3}{2}}\ , \quad \dots(77)$$

After simplifying and equating the like power of *x*, we get

$$c_0 = -1\ ,\ \ c_1 = 3\ , c_2 = 2, \quad \dots(78)$$

Substitute Eq. (78) in Eq. (74) we get,

$$u_0(x) = -1 + 3x + 2x^{1.5}\ , \quad \dots(79)$$

Hence,

$$u(x) = u_0(x) = -1 + 3x + 2x^{1.5}, \quad \dots(80)$$

Which the exact solution for Eq. (74) [9, 19],

Example 5:

Consider the following linear the WSVIE given in [19]:

$$u(x) = \frac{35}{33}x^{1.5} + \frac{5}{4}x + \frac{2}{3} + x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}}u(t)d\ , \quad \dots(81)$$

Where $f(x) = \frac{35}{33}x^{1.5} + \frac{5}{4}x + \frac{2}{3}$, $\square = \frac{1}{3}$, and $u(x) = \frac{7}{3}x^{1.5} + 5x - \frac{1}{3}$ is the exact solution.
Suppose the solution of Eq. (81).

$$u_0(x) = c_0 + c_1\boldsymbol{x} + c_2x^{\frac{3}{2}}\ , \quad \dots(82)$$

Substitute Eq. (82) into Eq. (81) we have,

$$c_0 + c_1\boldsymbol{x} + c_2x^{\frac{3}{2}} = \frac{35}{33}x^{\frac{3}{2}} + \frac{5}{4}x + \frac{2}{3} + x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}}(c_0 + c_1t + c_2t^{\frac{3}{2}})dt\ \ , \quad \dots(83)$$

After simplifying, we get,

$$c_0 + c_1\boldsymbol{x} + c_2x^{\frac{3}{2}} = \frac{35}{33}x^{\frac{3}{2}} + \frac{5}{4}x + \frac{2}{3}\ + 3c_0 + \frac{3}{4}c_1x + \frac{6}{11}c_2x^{\frac{3}{2}}\ , \quad \dots(84)$$

$$c_0 = -\frac{1}{3}\ ,\ \ c_1 = 5\ , \qquad c_2 = \frac{7}{3}, \quad \dots(85)$$

Substitute Eq. (85) in Eq. (82) we get,

$$u_0(x) = \frac{7}{3}x^{1.5} + 5x - \frac{1}{3}\ , \quad \dots(86)$$

Hence,

$$u(x) = u_0(x) = \frac{7}{3}x^{1.5} + 5x - \frac{1}{3}, \quad \dots(87)$$

Which the exact solution for Eq. (81) [9, 12],

Example 6:

Let us consider the WSVIE [20]:

$$u\ (x) = x(1-x) + \frac{16}{105}x^{7/2}(7-6x) - \int_0^x \frac{xt}{\sqrt{x-t}}u(t)dt\ \ , \quad \dots(88)$$

Where, $\mu = \frac{1}{2}, f(x) = x(1-x) + \frac{16}{105}x^{7/2}(7-6x)$, and the exact solution is $u\,(x) = x(1-x)$.
Now apply the MPSM to solving Eq. (88), we suppose the solution of Eq. (88)

$$u_0(x) = c_1x - c_2x^2 + c_3x^{7/2} + c_4x^{\frac{9}{2}}, \quad \ldots (89)$$

Substitute Eq. (89) into Eq. (88) we have,

$$c_1x - c_2x^2 + c_3x^{7/2} + c_4x^{\frac{9}{2}} = x - x^2 + \frac{16x^{7/2}}{15} - \frac{32x^{9/2}}{35} + x^{\frac{-1}{2}}\int_0^x t^{\frac{-1}{2}}\left(c_1t - c_2t^2 + c_3t^{7/2} + c_4t^{\frac{9}{2}}\right)dt\,, \quad \ldots (90)$$

By integrating and solving we get,

$$c_1x - c_2x^2 + c_3x^{7/2} - c_4x^{9/2} = x - x^2 + \frac{16x^{7/2}}{15} - \frac{16\,c_1}{15}x^{7/2} - \frac{32x^{9/2}}{35} + \frac{32}{35}c_2\,x^{9/2} - c_3\frac{63}{256}\pi x^6 + c_4\frac{231\pi x^7}{1024}, \quad \ldots (91)$$

After simplifying and equating the like power of *x*, we get the values of $c_1. c_2, c_3, c_4$.

$$c_1 = c_2 = 1\,, \qquad c_3 = c_4 = 0, \quad \ldots (92)$$

Substitute Eq. (92) in Eq. (89) we get,

$$u_0(x) = x(1-x)\,, \quad \ldots (93)$$

Hence,

$$u(x) = u_0(x) = x(1-x), \quad \ldots (94)$$

Which the exact solution for Eq. (89) [9, 20],

5.2 NONLINEAR WSVIE

In this subsection we will use the MPSM to handle the nonlinear WSVIE.

Case 1: $\mu > 1$:

We will discuss some examples of the nonlinear WSVIE when $\mu > 1$, and will be solved by proposed method to obtain the exact solution.

Example 7:

Consider the following nonlinear WSVIE of second kind [19]:

$$u\,(x) = \sqrt{x} - \frac{5}{11}x + x^{-\frac{12}{10}}\int_0^x t^{\frac{2}{10}}\,u^2(t)dt\ , \quad \ldots (95)$$

Where, $\mu = \frac{12}{10}, f(x) = \sqrt{x} - \frac{5}{11}x$, and the exact solution is $u\,(x) = \sqrt{x}$.
By using the proposed method, we suppose the solution of Eq. (95).

$$u_0(x) = c_1x^{\frac{1}{2}} - c_2x, \quad \ldots (96)$$

Substitute Eq. (96) into Eq. (95) we have,

$$c_1x^{\frac{1}{2}} - c_2x = \sqrt{x} - \frac{5}{11}x + x^{-\frac{12}{10}}\int_0^x t^{\frac{2}{10}}\left(c_1t^{\frac{1}{2}} - c_2t\right)^2 dt\ , \quad \ldots (97)$$

By integrating and solving we get,

$$c_1 x^{\frac{1}{2}} - c_2 x = \sqrt{x} - \frac{5}{11}x + c_1^2 \frac{5x}{11} - \frac{100}{27} c_1 c_2 x^{\frac{3}{2}} + c_2^2 \frac{125x^2}{16}, \quad \dots (98)$$

After simplifying and equating the like power of x, we get,

$$c_1 = 1, \qquad c_2 = 0, \quad \dots (99)$$

Substitute Eq. (99) in Eq. (96) we get,

$$u_0(x) = \sqrt{x}, \quad \dots (100)$$

Therefore,

$$u(x) = u_0(x) = \sqrt{x}, \quad \dots (101)$$

Which the exact solution for Eq. (95) [9, 19],

Example 8:

We consider the nonlinear WSVIE

$$u(x) = x - \frac{2x^3}{9} + x^{-\frac{15}{10}} \int_0^x t^{\frac{1}{2}} u^3(t)dt, \quad \dots (102)$$

Where, $\mu = \frac{15}{10}, f(x) = x - \frac{2x^3}{9}$, and the exact solution is $u(x) = x$.

Suppose the solution of Eq. (102).

$$u_0(x) = c_1 x - c_2 x^3, \quad \dots (103)$$

Substitute Eq. (103) into Eq. (102) we have,

$$c_1 x - c_2 x^3 = x - \frac{2x^3}{9} + x^{-\frac{15}{10}} \int_0^x t^{\frac{1}{2}} (c_1 t - c_2 t^3)^3 dt, \quad \dots (104)$$

By integrating and solving we get,

$$c_1 x - c_2 x^3 = x - \frac{2x^3}{9} + \frac{2{c_1}^3 x^3}{9} - \frac{6}{13}{c_1}^2 c_2 x^5 + \frac{6}{17} c_1 {c_2}^2 x^7 - \frac{2{c_2}^3 x^9}{21}, \quad \dots (105)$$

After equating the like power of x, we get the values of c_1, c_2.

$$c_1 = 1, \qquad c_2 = 0, \quad \dots (106)$$

Now substitute Eq. (106) in Eq. (103) we get,

$$u_0(x) = x, \quad \dots (107)$$

Therefore,

$$u(x) = u_0(x) = x, \quad \dots (108)$$

Which the exact solution for Eq. (102) [9, 19],

Case 2: $0 < \mu \leq 1$:

We applied the proposed method on equations of the nonlinear WSVIE when $0 < \mu \leq 1$.

Example 9:

Consider the nonlinear WSVIE [21]

$$u(x) = x^{1/2} + \frac{3\pi}{8}x^2 + \int_0^x -\frac{1}{(x-t)^{\frac{1}{2}}} u^3(t)dt, \quad \dots (109)$$

Let $\mu = \frac{1}{2}$, $f(x) = x^{1/2} + \frac{3\pi}{8}x^2$, and the exact solution is $u(x) = \sqrt{x}$.

By using MPSM, we assume the solution of Eq. (109).

$$u_0(x) = c_1 x^{\frac{1}{2}} + c_2 x^2, \quad \dots (110)$$

Now Substitute Eq. (110) into Eq. (109) we have,

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$$c_1 x^{\frac{1}{2}} + c_2 x^2 = x^{1/2} + \frac{3\pi}{8} x^2 + \int_0^x - \frac{1}{(x-t)^{\frac{1}{2}}} u^3(t)dt \ , \qquad \dots(111)$$

By integrating and solving we get,

$$c_1 x^{\frac{1}{2}} + c_2 x^2 = x^{1/2} + \frac{3\pi}{8} x^2 - \frac{3}{8} c_1{}^3 \pi x^2 - \frac{96}{35} c_1{}^2 c_2 \, x^{7/2} - \frac{189}{256} c_1 c_2{}^2 \pi x^5 - \frac{2048 c_2{}^3 x^{13/2}}{3003}, \qquad \dots(112)$$

By equating the like power of x, we get the values of c_1, c_2.

$$c_1 = 1 \,, \qquad c_2 = 0, \qquad \dots(113)$$

Now substitute Eq. (113) in Eq. (110) we get,

$$u_0(x) = x^{\frac{1}{2}}, \qquad \dots(114)$$

Therefore,

$$u(x) = u_0(x) = x^{\frac{1}{2}}, \qquad \dots(115)$$

Which is the exact solution of integral equation (109) [21].

Example 10:

Consider the following nonlinear WSVIE [21]

$$u_1\,(x) = x^{1/2} - \frac{16}{15} x^{5/2} + \int_0^x \frac{u^4(t)}{(x-t)^{\frac{1}{2}}} dt, \qquad \dots(116)$$

Where $\mu = \frac{1}{2}, f(x) = x^{1/2} - \frac{16}{15} x^{5/2}$, and the exact solution is $u\,(x) = \sqrt{x}$.

Now apply the MPSM to solving Eq. (116), we suppose the solution of Eq. (116)

$$u_0(x) = c_1 x^{\frac{1}{2}} - c_2 x^{\frac{5}{2}}, \qquad \dots(117)$$

Substitute Eq. (117) into Eq. (116) we have,

$$c_1 x^{\frac{1}{2}} - c_2 x^{\frac{5}{2}} = x^{1/2} - \frac{16}{15} x^{5/2} + \int_0^x \frac{u^4(t)}{(x-t)^{\frac{1}{2}}} dt, \qquad \dots(118)$$

By integrating and solving we get,

$$c_1 x^{\frac{1}{2}} - c_2 x^{\frac{5}{2}} = x^{1/2} - \frac{16}{15} x^{5/2} + \frac{16}{15} c_1{}^4 x^{5/2} - \frac{1024}{315} c_1{}^3 c_2 x^{9/2} + \frac{4096 c_1{}^2 c_2{}^2 x^{13/2}}{1001} - \frac{262144 c_1 \, c_2{}^3 x^{17/2}}{109395} + \frac{524288 c_2{}^4 x^{21/2}}{969969}, \qquad \dots(119)$$

After equating the like power of x, we get the values of c_1, c_2.

$$c_1 = 1 \,, \qquad c_2 = 0, \qquad \dots(120)$$

Now substitute Eq. (120) in Eq. (117) we get,

$$u_0(x) = x^{\frac{1}{2}}, \qquad \dots(121)$$

Therefore,

$$u(x) = u_0(x) = x^{\frac{1}{2}}, \qquad \dots(122)$$

Which is the exact solution of integral equation (116) [21].

CONCLUSION

In this paper, we implement the modified power series method to obtain the exact solutions after solving linear and nonlinear WSVIE in two cases: case 1: □ > 1 and case 2: $0 < \mu \leq 1$. The results showed that the method is easy to implement and it converge to the exact solution if it exist. In this method the solution is obtained in the series form that converges to the exact solution with easily computed components. The method gives rapid convergent and can be easily comprehended with only a basic knowledge of Calculus. It is economical in terms of computer power/memory and does not involve tedious calculations. The software used for the calculations in this study was MATHEMATICA® 9.0.

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