Small-Closed Submodules

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ABSTRACT

Throughout this paper R represents an associative ring with identity and M is unitary right R-module, the purpose of this paper is to study a new concept, named s-closed submodules. It is stronger than the concept of closed submodules, where a submodule N of an R-module M is called s-closed in M, if it has no proper small-essential in M. As an application of this closedness, we consider s-closed injective modules.

Key words: Small submodules, Essential submodules, Small-essential submodules, closed submodules, s-closed submodules, Hollow modules and s-closed injective modules.

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1. INTRODUCTION

Let R be an associative ring with identity and M a unitary right R-module. It is well known that a submodule K of an R-module M is said to be small in M notationally, $K \leq_s M$, if $K + L = M$ for every submodule L of M, then $L = M$. Dually, a nonzero submodule K of M is called essential (briefly $K \leq_e M$), if $K \cap L \neq 0$ for each nonzero submodule L of M [6, P.106], if this is case, then we say that M is essential extension of K. Let K be a submodule of a module M. K is said to be small-essential (shortly s-essential) in M and denoted by $K \leq_{se} M$, if $K \cap L = 0$ with $L \leq_s M$ implies $L = 0$. Equivalently, $K \leq_{se} M$ if and only if for each $0 \neq m \in M$, if $Rm \leq_s M$, then there is an element $r \in R$ such that $0 \neq rm \in K$ [9].

A submodule N of an R-module M is called closed in M, if it has no proper essential extension in M [5, P.18]. In our work we introduce a new class of closed submodules, namely small-closed submodules, which is stronger than that of closed submodules, where a submodule K of an R-module M is called small-closed (simply s-closed) if K has no proper small-essential extension in M, i.e. if $K \leq_{se} L \leq M$ then $K = L$.

Following C. Gomes [5], let M_1 and M_2 be an R-modules, M_1 is called M_2-C-injective if every homomorphism $\alpha: K \longrightarrow M_1$, where K is a closed submodule of M_2, can be extended to a homomorphism β: $M_2 \longrightarrow M_1$. This concept lead us to introduced the following: The module M_1 is called scl-M_2-injective if every homomorphism α: K ⟶ M_1, where K is a s-closed submodule of M_1, can be extended to a homomorphism β: $M_1 \longrightarrow M_2$. This paper consist of two sections, in section one we investigate the main properties of s-closed submodules. Also we study the relationships between s-closed submodules, closed submodules and y-closed submodules [5, P.42]. In section two, we introduced the concept s-closed injective and study some basic properties.

2. Small-Closed Submodules

Definition (2.1): A submodule K of an R-module M is called s-closed, denoted by $K \leq_{sc} M$, if K has no proper s-essential extension in M, that is, whenever $N \leq M$ such that $K \leq_{se} N \leq M$, then $K = N$. An right ideal I of R is called an s-closed, if it is s-closed submodule in R_R.

Remarks and Examples (2.2):

1. Every s-closed submodule in R-module M is closed in M.

Proof: Let K be an s-closed submodule in M, and let $K \leq M$ with $K \leq_{e} N \leq M$. By [9], then $K \leq_{se} N$. But K is an s-closed submodule in M, thus K = N.

The converse may not true in general, for example: Consider Z_{12} as Z-module, let $4Z_{12} \leq_c Z_{12}$ but $4Z_{12} \nleq_{sc} Z_{12}$ since $4Z_{12} \leq_{se} 2Z_{12}$.

2. It is well-known that, every direct summand is closed by [5], but in case s-closed submodules, there is no relationship with direct summands. For example: Consider Z_6 as Z-module, the non-trivial direct summands of Z_6 are $2Z_6$and $3Z_6$ which aren't s-closed submodule of Z_6 since $2Z_6 \leq_{se} Z_6$ and $3Z_6 \leq_{se} Z_6$.

3. The intersection of s-closed submodules of an R-module M is s-closed in M.

Proof: Let $H \leq M$ such that $A \cap B \leq_{se} H \leq M$. By [9], $A \leq_{se} H$ and $B \leq_{se} H$. Since A and B are s-closed submodules in M, then A = H = B, hence $A \cap B = H$.

4. Let M be an R-module, and A an s-closed submodule of M. If B is a submodule of M such that $A \cong B$, then it is not necessary that B is an s-closed submodule in M. For example, consider Z as Z-module, Z is an s-closed submodule in Z, and $Z \cong 2Z$, but 2Z is not s-closed submodule in Z, since 2Z is an s-essential submodule of Z.

5. Let A and B be submodules of an R-module M such that $A \leq B \leq M$. If B is s-closed submodule in M, then A need not be s-closed submodule in M. For example; consider Z as Z-module, Z is an s-closed submodule of Z and $3Z \leq Z$, while 3Z is not s-closed submodule in Z.

6. Let $K \leq Q$ be submodules of M with K is s-closed in M. Then $Q \leq_{se} M$ if and only if $\frac{Q}{K} \leq_{se} \frac{M}{K}$.

Proof: Suppose that $\frac{P}{K} \leq_{s} \frac{M}{K}$ such that $\frac{Q}{K} \cap \frac{P}{K} = 0$ and since $Q \leq_{se} M$ and $P \leq_{se} P$ so $K = Q \cap P \leq_{se} P$ and since $K \leq_{sc} M$ so $K = P$ i.e. $\frac{P}{K} = 0$. The converse is clear by [9].

Issue 6 volume 1, Jan. –Feb. 2016
ISSN 2249-9954

Recall that a nonzero R-module M is hollow if every proper submodule of M is small [3].

Remark (2.3): In case of hollow modules, the concept of essential submodule and s-essential are equivalent and hence closed and s-closed are equivalent.

Recall that a singular submodule defined by Z(M) = { m ∈ M : $r_R(m) \leq_e R$ } equivalent, Z(M) is the set of those m ∈ M for which mI =0, for some essential right ideal I of R. Z(M) is a submodule of M, it is called the singular submodule of M. An R-module M is called a singular if Z(M) = M. At the extreme, M is called nonsingular if Z (M) = 0 [5]. A submodule N of M is a y-closed submodule of M provided M/N is nonsingular [5].We cannot find a direct relation between s-closed and y-closed submodules. However, under some conditions we can find some cases of this relationship as the following theorem shows.

Proposition (2.4): Let M be a hollow R-module, and let K be a submodule of M. Consider the following statements:

1. K is an s-closed submodule;
2. K is an closed submodule;
3. K is an y-closed submodule.

Then (1) ⇔ (2) and (3) ⇒ (2). If M is a nonsingular module, then (1) ⇒ (3).

Proof: Since M is hollow module then by Remark (2.3) and by Remarks and Examples (2.2), then (1) ⇔ (2). And (3) ⇒ (2) By [8]. If M is a nonsingular module, then (1) ⇒ (3) Let K be an s-closed submodule in M, By Remarks and Examples (1.2), K be a closed submodule in M. But M is a nonsingular module, so by [8], N is a y-closed submodule of M. □

Proposition (2.5): Let M_1 and M_2 be two R-modules. Then A is an s-closed submodule in M_1 and B is an s-closed submodule in M_2 if and only if A⊕B is an s-closed submodule in $M = M_1 \oplus M_2$.

Proof: Let $A \oplus B \leq_{se} N$. Consider π_i: M⟶ M_i (i=1, 2) the projection maps. Clear that by [1] $N = \pi_1(N) \oplus \pi_2(N)$ since π_1 and π_2 are an idempotent. Hence $A \oplus B \leq_{se} N = \pi_1(N) \oplus \pi_2(N)$, and by [9] we get $A \leq_{se} \pi_1(N)$ and $B \leq_{se} \pi_2(N)$. But A and B are s-closed, so $A = \pi_1(N)$ and $B = \pi_2(N)$. Thus $A \oplus B \leq_{sc} M = M_1 \oplus M_2$. Conversely, Let $A \leq_{se} K_1$ and $B \leq_{se} K_2$, then by [9] $A \oplus B \leq_{se} K_1 \oplus K_2$ but $A \oplus B \leq_{sc} M = M_1 \oplus M_2$, thus $A \oplus B = K_1 \oplus K_2$ so $A = K_1$ and $B = K_2$. □

Proposition (2.6): Let f: M ⟶ M' be an epimorphism and N a submodule of M such that ker f ⊆ N. If N is an s-closed in M, then f (N) is an s-closed in M'.

Proof: Let K' be a submodule of M' such that $f(N) \leq_{se} K' \leq M'$. Then $f^{-1}(f(N)) \leq_{se} f^{-1}(K') \leq M$ by [9]. We can easily show that $f^{-1}(f(N)) = N$ since ker f ⊆ N. This implies that $N \leq_{se} f^{-1}(K')$.

But N is an s-closed submodule in M, then N = f^{-1} (K'). Since f is epimorphism so f(N) = $f(f^{-1}(K'))$ = K'⋂ M' = K'. □

Corollary (2.7): Let A and B be submodules of an R-module M. If B is an s-closed submodule in M, then B/A is an s-closed submodule in M/A. □

Proposition (2.8): Let M be an R-module, and let A and B be submodules of M such that A ≤ B. If A is an s-closed submodule in M, then A is an s-closed submodule in B.

Proof: Suppose that $A \leq_{se} L \leq B$, so L ≤ M. But A is an s-closed submodule in M, therefore A = L. □

Corollary (2.9): Let A and B be submodules of an R-module M. If A∩B is an s-closed submodule in M, then A ∩ B is an s-closed submodule in A and B. □

Corollary (2.10): If N and K are s-closed submodules in an R-module M, then N and K are s-closed submodules in N + K.

Recall that an R-module M is called chained if for each submodules A and B of M either A ≤ B or B ≤ A [7].

Proposition (2.11): Let M be a chain R-module, and let A and B be submodules of M such that A ≤ B ≤ M. If A is an s-closed submodule in B and B is an s-closed submodule in M then A is an s-closed submodule in M.

Proof: Let K be a submodule of M such that $A \leq_{se} K \leq M$. By assumption we have two cases: If K ≤ B, since A is an s-closed submodule in B then A = K, hence A is an s-closed submodule in M. If B ≤ K, since $A \leq_{se} K$, so by [9], $B \leq_{se} K$. But B is an s-closed in M, thus B = K. That is $A \leq_{se} B$. On the other hand, A is an s-closed submodule in B, so A = B, hence A is an s-closed submodule in M. □

Consider the following condition(∗): Every small submodule of an R-module M is small in every it is (proper) extension in M, that is, if K ≤ L ≤ M, where K and L are submodules of M, if $K \leq_s M$, then $K \leq_s L$.

There is a question will arises, is there an R-module which satisfies condition(∗)?

From [2, Pro. (20.2)] the condition(∗) holds in every supplemented module [2, P.236].

Now, we will use condition(∗) to prove the following propositions:

Proposition (2.12): Let an R-module M with condition($*$) and let $K \leq L \leq M$. Then $K \leq_{se} L$ and $L \leq_{se} M$, if and only if $K \leq_{se} M$.

Proof: ⇒) Let N be a small submodule of M with $K \cap N = 0$. Then we have two cases:

i. If $N \leq L$, then by condition($*$) $N \leq_s L$ and since $K \leq_{se} L$, then $N = 0$.

ii. If $N \nleq L$, then $N \cap L \leq N \leq M$ and hence $N \cap L \leq_s M$, by condition($*$) $N \cap L \leq_s L$ thus $K \cap N \cap L = K \cap N = 0$, but $K \leq_{se} L$, then $N \cap L = 0$ and since $L \leq_{se} M$, then $N = 0$. This shows that $K \leq_{se} M$.

⇐) It is clear by [9, Pro.2.7].

Note (2.13): The result above is incorrect without condition ($*$) see [9, Ex.2.8].

Proposition (2.14): Let M be an R-module with condition($*$). Then every submodule of M is s-essential in s-closed.

Proof: Let C be a submodule of an R-module M. Consider the collection $\mathbb{C} = \{K : K \leq M \mid C \leq_{se} K\}$. It is clear that $\mathbb{C}$ is nonempty and $\mathbb{C}$ is partial order set with respect to inclusion relation of submodules. Let $\mathbb{C}' = \{ K_\alpha \mid K_\alpha \in \mathbb{C}, \alpha \in \Lambda \}$ be a chain in $\mathbb{C}$ and let $W = \bigcup_{\alpha \in \Lambda} K_\alpha$, to show that $C \leq_{se} W$. For each $x \in W$ with $xR \leq_s W$ then $xR \leq K_{\alpha_0}$ for some $\alpha_0 \in \Lambda$, so by condition($*$), $xR \leq_s K_{\alpha_0}$, then there exist $r(\neq 0) \in R$ and $xr(\neq 0) \in K_{\alpha_0} \leq W$. Therefore, $W \in \mathbb{C}$. Hence $\mathbb{C}'$ has an upper bound W in $\mathbb{C}$. By Zorn's lemma $\mathbb{C}$ has a maximal element say H, to show that H is s-closed in M. Let D be a submodule of M such that $H \leq_{se} D$ and since H is maximal submodule, thus $H = D$. □

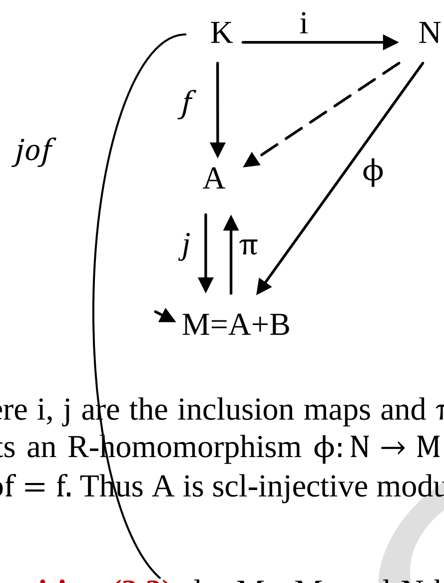
3. Injectivity on Small-Closed Submodules

This section we give a generalization of injective, namely s-closed injective (shortly scl-injective)

Definition (3.1): Let M_1 and M_2 be an R-modules. M_2 is called scl-M_1-injective if every homomorphism $\alpha: K \longrightarrow M_2$, where K is a s-closed submodule of M_1, can be extended to a homomorphism $\beta: M_1 \longrightarrow M_2$. An R-module M is called quasi-scl-injective module if M is scl-M-injective. An R-module M is called scl-injective if M is scl-N-injective, for every R-module N. Clearly, if M_2 is M_1-injective, then M_2 is scl-M_1-injective and the converse is not true in general, for example: Consider the module Z_n as Z-module. Since Z_n is the only s-closed submodule of Z_n and hence Z_n is scl-injective. But Z_n is not divisible where $0 = n Z_n \neq Z_n$. Hence Z_n is not injective.

Proposition (3.2): A direct summand of scl-N-injective module is scl-N-injective.

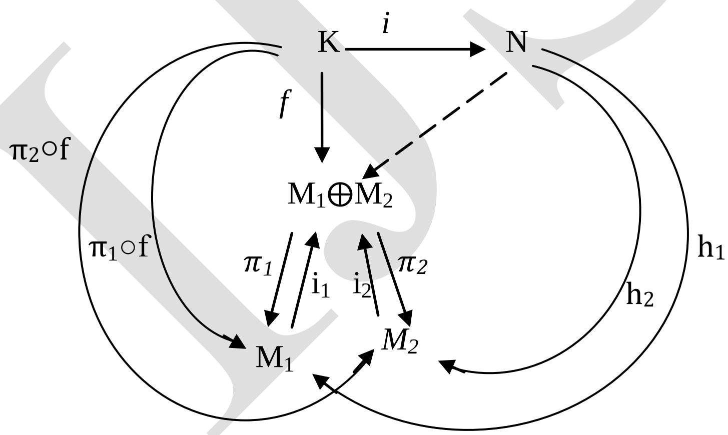
Proof: let $M = A \oplus B$ be scl-N-injective. To show that A is scl-N-injective. Let K be an s-closed submodule of N and let $f: K \longrightarrow A$ be an R-homomorphism. Now consider the following diagram



Where i, j are the inclusion maps and π is the projection map since M is scl-injective, then there exists an R-homomorphism $\phi: N \to M$ such that $jof = \phi oi$. Let $g = \pi o \phi$, then $goi = \pi o \phi oi = \pi ojof = f$. Thus A is scl-injective module. □

Proposition (3.3): let M_1, M_2 and N be R-modules. If M_1 and M_2 are scl-N-injective modules, then $M_1 \oplus M_2$ is scl-N-injective.

Proof: Suppose that M_1 and M_2 are scl-N-injective modules. Let K be an s-closed submodule of N and let $f: K \longrightarrow M_1 \oplus M_2$ be an R-homomorphism. Now consider the following diagram



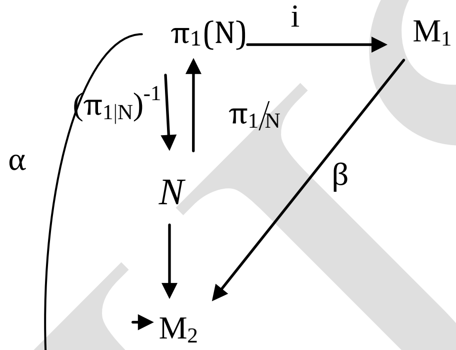
Where i_1, i_2 are the inclusion maps and π_1, π_2 are the projection map. Since M_1 and M_2 are scl-N-injective modules, then there exists homomorphism $h_1: N \longrightarrow M_1$ and $h_2: N \longrightarrow M_2$ such that $\pi_1 of = h_1 oi$ and $\pi_2 of = h_2 oi$. Define $h: N \longrightarrow M_1 \oplus M_2$ as follows $h(n) = (h_1(n), h_2(n))$, for each $n \in N$. To show that $f = hoi$. Let $k \in K$, then $f(k) = (m_1, m_2)$, where $m_1 \in M_1$ and $m_2 \in M_2$. Thus $hoi(k) = h(i(k)) = (h_1(i(k)), h_2(i(k))) = (\pi_1 of(k), \pi_2 of(k)) = (m_1, m_2)$. Hence $M_1 \oplus M_2$ is scl-N-injective module. □

Proposition (3.4): Let M be an R-module, and let A be an s-closed submodule of M. If A is scl-M-injective module, then A is a direct summand of M.

Proof: Assume that A is scl-M-injective module and let $I: A \longrightarrow A$ be the identity map since A is scl-M-injective module, then there exists a homomorphism $f: M \longrightarrow A$ such that $I = f \circ i$. Claim that $M = \ker f \oplus A$. Let $x \in M$, then one can easily show that $x - f(x) \in \ker f$, thus $x = f(x) + x - f(x) \in A + \ker f$. To show that $A \cap \ker f = 0$, let $x \in A \cap \ker f$, so $f(x) = 0$. But $f(x) = x$, therefore $x = 0$. Thus A is a direct summand of M. □

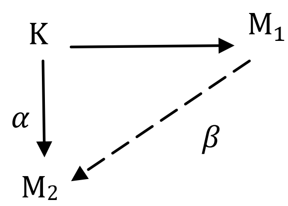
Theorem (3.5): Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. Then M_2 is scl-M_1-injective if and only if, for every submodule N of M such that $N \cap M_2 = 0$ and $\pi_1(N)$ is a s-closed submodule of M_1, there exists a submodule N′ of M such that $N \leq N'$ and $M = N' \oplus M_2$, where π_1 the natural projection of M into M_1.

Proof: Assume that M_2 is scl-M_1-injective. Let $\pi_i: M \longrightarrow M_i$ (i=1,2) denote the projection mapping and let N be a submodule of M such that $N \cap M_2 = 0$ and $\pi_1(N) \leq_{sc1} M_1$. Consider the following diagram



Define $\alpha: \pi_1(N) \longrightarrow M_2$ as follows, for every $x \in \pi_1(N)$ there is $n \in N$ such that $x = \pi_1(n)$ set $\alpha(x) = \pi_2(n)$. Clearly, α is well-define and homomorphism, the map α can be extended to a homomorphism $\beta: M_1 \longrightarrow M_2$ since M_2 is scl-M_1-injective and $\pi_1(N) \leq_{sc} M_1$. Define $N' = \{x + \beta(x) \mid x \in M_1\}$, clearly N' is a submodule of M and claim $M = N' \oplus M_2$. To see this, let $m \in M$, there exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2 = (m_1 + \beta(m_1)) + m_2 - \beta(m_1) \in N' \oplus M_2$. Now suppose that $m \in N' \cap M_2$, then $m = m_2 = m_1 + \beta(m_1)$, where $m_1 \in M_1$ and $m_2 \in M_2$. Hence $m_1 = m_2 - \beta(m_1) \in M_1 \cap M_2 = 0$, so $N' \cap M_2 = 0$. Thus $M = N' \oplus M_2$. Now let $0 \neq x = a + b \in N$, where $a \in M_1$, $b \in M_2$, since $\pi_1(x) = a$ and $\pi_2(x) = b$, then $x = \pi_1(x) + \pi_2(x) = \pi_1(x) + \beta\pi_1(x) \in N'$ since for every $x \in N$, $\beta\pi_1(x) = \alpha\,\pi_1(x) = \pi_2(x)$. Thus N is a submodule of N'. Conversely; assume that, for every submodule N of M such that $N \cap M_2 = 0$ and $\pi_1(N) \leq_{sc} M_1$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let K be a s-closed[1] submodule of M_1 and let $\alpha: K \longrightarrow M_2$ be a homomorphism. Define $N = \{x - \alpha(x) \mid x \in K\}$, clearly N is a submodule of M. Claim that $N \cap M_2 = 0$, to see this, let $x' \in N \cap M_2$, then $x' \in M_2$ and $x' = x - \alpha(x)$, $x \in K$, hence $x' + \alpha(x) = x \in M_1 \cap M_2$ which means that

$x' = 0$. Thus $N \cap M_2 = 0$. It is not hard to prove that $\pi_1(N) = K$ and so $\pi_1(N) \leq_{sc} M_1$. Then, by hypothesis, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let $\pi_2: M \longrightarrow M_2$ denote the projection with kernel N' and let $\beta: M_1 \longrightarrow M_2$ be the restriction of π to M_1. Consider the following diagram.



For every $x \in K$, $\beta(x) = \pi(x) = \pi(x - \alpha(x) + \alpha(x)) = \alpha(x)$ and therefore, β extends α. Thus M_2 is scl-M_1-injective. □

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