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Extending and P-extending S-act over Monoids

Shaymaa Amer

Department of Mathematics, College of Basic Education , Mustansiriyah University, Baghdad, Iraq.

ABSTRACT

Let S be a monoid with zero element . In this work we introduce and study some properties of extending and P-extending S-act over Monoids . Some results on extending and P-extending modules are extended to those acts . Some new properties of extending and P-extending acts are considered and obtained .

Keywords : Extending acts , P-extending acts , Fully invariant subacts , continuous acts , quasi acts, ⋂-large cyclic extending act .

Corresponding Author: Shaymaa Amer E MAIL *shaymaa_amer76@yahoo.com*

1-INTRODUCTION

Throughout this paper , S is represents a monoid with zero elements 0 and every right S-act is a unitary with zero elements Θ which denoted by M_s . A right S-act M_s with zero is a non-empty set with a function f : $M\times S\rightarrow M$, $f(m,s)\mapsto ms$ such that the following properties hold : (1) $m\cdot 1= m$ (2) $m(st)= (ms)t$, for all $m \in M$ and s , t ∈ S, 1 is the identity element of S .

An S-act B_s is a retract of S-act A_s if and only if there exists a subact W of A_s and S-epimorphism $f:A_s\rightarrow W$ such that $B_s \simeq W$ and f(w) = w for every w ∈ W [6,P.84] . An S-homomorphism f which maps an S-act M_s into an S-act N_s is said to be split if there exists S-homomorphism g which maps N_s into M_s such that $fg = 1_N$ [5] .

A subact N of M_s is called **large** (or **essential**) in M_s if and only if any homomorphism $f:M_s\rightarrow H_s$, where H_s is any S-act with restriction to N is one to one ,then f is itself one to one [3].In this case we say that M_s is essential extension of N . In [3], Berthiaume showed that every S-act has a maximal essential extension which is injective and it is unique up to S-isomorphism over M_s . A non-zero subact N of M_s is **intersection large** if for all non-zero subact A of M_s , $A \cap N \neq \Theta$, and will denoted by N is ⋂-large in M_s[12] .

In [4] , Feller and Gantos proved that every large subact of M_s is ⋂-large, but the converse is not true in general . An equivalence relation ρ on a right S-act M_s is a **congruence** relation if aρb implies that asρbs for all a,b∈ M_s and s ∈ S [7].The congruence ψ_M is called **singular** on M_s and it is defined by $a\psi_M b$ if and only if ax = bx for all x in some ⋂-large right ideal of S [8] . A non-zero S-act M_s over a monoid S is called **reversible** (**⋂-reversible**) if every non-zero subact of M_s is large (⋂-large) , it is clear that every nonzero reversible act is ⋂-reversible act , but the converse is not true in general and they are coincide when ψ_M = i [2] . An element s ∈ S is called **left** (**right**) **cancellable** if sr = st (rs = ts) for r,t∈ S implies r = t and cancellable if s is left and right cancellable . The semigroup S is called **cancellative** if all elements of S are cancellable ([6],P.30) .

An S-act A_s is called cyclic (or principal) act if it is generated by one element and it is denoted by A_s= <u> where u ∈ A_s ,then A_s= uS ([6],P.63) . An S-act M_s is called decomposable if there exist two subacts A , B

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of M_s such that $M_s = A \cup B$ and $A \cap B = \Theta$. In this case , $A \cup B$ is called decomposition of M_s . Otherwise M_s is called indecomposable([6],p.65) . Every cyclic act is indecomposable .

An S-act M_s is called simple if it contains no subact other than M_s itself . An S-act M_s is called Θ-simple if it contains no subacts other than M_s and one element subact Θ_s ([10], p.13] .

A subact N of a right S-act M_s is called fully invariant if $f(N) \subseteq N$ for every endomorphism f of M_s and M_s is called duo if every subact of M_s is fully invariant [9]

Let A_s , M_s be two S-acts . A_s is called M-injective if given an S-monomorphism $\alpha : N \to M_s$ where N is a subact of M_s and every S-homomorphism $\beta : N \to A_s$, can be extended to an S-homomorphism $\sigma : M_s \to A_s$. An S-act A_s is injective if and only if it is M-injective for all S-acts M_s [13] . An S-act A_s is quasi injective if and only if it is A-injective . Quasi injective S-acts have been studied by Lopez and Luedeman [7] .

An S-act A_s is called principally injective if it is M-injective to embeddings of all principal (cyclic) subact N into M_s .

A subact N of S-act M_s is called closed if it has no proper ∩-large extension in M_s that is the only solution of $N \hookrightarrow^{\cap l} L \underset{\neq}{\hookrightarrow} M_s$ is N = L . An S-act M_s is said to be satisfy C_2-condition if every subact of M_s which is isomorphic to retract of M_s is itself a retract of M_s [11] .

In this paper , we adopt another generalization of quasi injective and principally injective act which are extending and principally extending acts .

2- Extending Acts :

Definition(2.1): An S-act M_s is called extending act (for short CS-act) if every subact of M_s is ∩-large in a retract of M_s .

Definition(2.2): An S-act M_s is called semisimple if and only if every subact of M_s is a retract or it is union of simple subacts .

Examples and Remarks(2.3) :

1- Every quasi injective (injective) act is extending , but the converse is not true in general , for example Z with multiplication as Z-act is extending but not quasi injective , assume that Z is quasi-injective Z-act and let $f: 2Z \longrightarrow Z$ be the Z-homomorphism defined by $f(2n) = n$ for each $n \in Z$. Then , there is an endomorphism $g: Z \longrightarrow Z$ such that g extends f , thus $n = f(2n) = g(2n) = 2n\, g(1)$ and hence $g(1)=1/2$ which is a contradiction . Therefore Z is not a quasi-injective Z-act.

2- Every semisimple act is extending (since every subact is a retract of its act), but the converse is not true in general for example Z with multiplication as Z-act .

3- Isomorphic act to extending is extending act .

Lemma(2.4) : Let A and B be two subacts of an S-act M_s . If A is closed in B and B is closed in M_s , then A is closed in M_s .

Proof : Assume that A is closed (maximal ∩-large) subact of B , where B is closed act of S-act M_s . Let $\mathbb{C}$ be the collection of the set of all proper subact of M_s which is ∩-large extension of A in M_s . $\mathbb{C} \neq \emptyset$, since $B \in \mathbb{C}$. By Zorn's lemma , there exists maximal ∩-large extension C of A in M_s , which is closed subact of M_s . Then , by maximality of A , we have A = C and A is closed subact of M_s . ■

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Propositon (2.5) : An S-act M_s is extending(CS) if and only if every closed subact of M_s is a retract of M_s .

Proof : $\Longrightarrow$) Assume that M_s is extending act . Let N be a closed subact of M_s , since M_s is extending , then there exists a retract H of M_s such that N is $\cap$-large in H . But N is closed in M_s . So , N = H . Hence , N is a retract of M_s .

$\Longleftarrow$) Let A be a subact of M_s . Thus, by Zorn's lemma , there exists a closed subact B of M_s such that A is $\cap$-large in B . Since B is closed in M_s , thus by hypothesis B is a retract of M_s . Therefore, A is $\cap$-large in a retract of M_s . Hence , M_s is extending act . ■

Proposition(2.6) : Let M_s be an S-act and N be a subact of M_s . If H is any relative complement for N in M_s . Then $N \dot{\cup} H$ is $\cap$-large subact of M_s .

Proof : Since H is the complement to N , so $N \cap H = \Theta$. Thus, we have $N \cup H = N\dot{\cup}H$. This implies that $N\dot{\cup}H$ is a subact of M_s . Assume that A is a subact of M_s with $A \cap (N \dot{\cup} H) = \Theta$. Thus, $(N \dot{\cup} H) \cup A$ is direct . This means that $(N \dot{\cup} H) \cup A = N \dot{\cup} H \dot{\cup} A$ is direct whence $N \cap (H \dot{\cup} A) = \Theta$. Since $N \cap H = \Theta$ and $N \cap (H \dot{\cup} A) = \Theta$ so , by maximality of H we have $H \dot{\cup} A = H$. Thus $A = \Theta$. Therefore, $N \dot{\cup} H$ is $\cap$-large subact of M_s . ■

Theorem (2.7): The following statements are equivalent for an S-act M_s :

1- M_s is CS-act ,

2- Every closed subact of M_s is a retract ,

3- If A is a retract of $E(M_s)$, then $A \cap M_s$ is a retract of M_s .

Proof : ($1\longrightarrow2$) By proposition(2.5) .

($2\longrightarrow3$) Let $E(M_s) = A \dot{\cup} B$, where B is a subact of $E(M_s)$. Suppose that $A\cap M_s$ is $\cap$-large in K , where K is a subact of M_s and then of $E(M_s)$ and let $k \in K$. Then $k \in E(M_s)$, which implies that $k \in A$ or $k \in B$ (this means that k = a or k = b) . Now , consider $k \notin A$ and $k = b \neq \Theta$. Since M_s is $\cap$-large in $E(M_s)$, so there exists $\Theta \neq s \in S$ such that $ks = bs \in M_s$. But , $\Theta \neq b \in B$ and , so $bs \in B$. Thus , we have $bs \in M_s \cap B$. On the other hand , we have $A\cap M_s$ is $\cap$-large in K and B is $\cap$-large in B , so $A\cap M_s\cap B$ is $\cap$-large in $K\cap B$. But $M_s\cap A\cap B = \Theta$, so $K\cap B = \Theta$ and then $bs = \Theta$ which is a contradiction . Therefore $A\cap M_s$ is closed of M_s and hence by (2) it is a retract of M_s .

($3\longrightarrow1$) Let A be a subact of M_s and B be a relative complement of A . Then , by lemma(2.6) $A\dot{\cup}B$ is $\cap$-large subact of M_s . As M_s is $\cap$-large in $E(M_s)$, so $A\dot{\cup}B$ is $\cap$-large in $E(M_s)$ by lemma(3.1) in [5] and so $E(A)\dot{\cup}E(B) = E(A\dot{\cup}B) = E(M_s)$. Since E(A) is a retract of $E(M_s)$, then by (3) $E(A)\cap M_s$ is a retract of M_s . But A is $\cap$-large in E(A) and M_s is $\cap$-large in M_s , then $A = A \cap M_s$ is $\cap$-large in $E(A)\cap M_s$. Thus , M_s is extending act . ■

The following propositions (2.8) , (2.9) and (2.10) explain under which the subact of extending act inherit this property :

Proposition (2.8): A closed subact of extending act is extending .

Proof: Let N be a closed subact of extending act M_s . Let A be a closed subact of N . By lemma(2.4) A is closed in M_s , but M_s is extending act , so by proposition(2.5) A is a retract of M_s and since A subact of N ,then A is a retract of N . Thus N is extending act.■

Proposition (2.9): Every subact N of extending act M_s with the property that intersection of N with any retract of M_s is retract of N , is extending .

Proof: Let N be subact of M_s and A be subact of N . Since M_s is extending act , so there exists a retract B of M_s such that A is $\cap$-large in B . But $A \subseteq B \cap N \subseteq B$, thus by lemma(3.1) in [5] , A is $\cap$-large in $B\cap N$ and by hypothesis $B\cap N$ is a retract of N. Hence N is extending.■

Proposition (2.10): Any fully invariant subact of extending S-act is extending .

Proof: Let M_s be extending S-act and N be fully invariant subact of M_s . If A is subact of N , then A is a subact of M_s . Since M_s is extending , so there exists a retract D of M_s such that A is $\cap$-large in D . This means that $M_s = D \dot{\cup} H$, where H any subact of M_s . As N is fully invariant , so $N = (N\cap D) \dot{\cup} (N\cap H)$. This implies that $N\cap D$ is a retract of N . Since A is $\cap$-large in D and N is $\cap$-large in N , so $A = N\cap A$ is $\cap$-large $N\cap D$. Hence , N is extending . ■

Proposition (2.11): Let $M_s = M_1 \dot{\cup} M_2$, where M_1 and M_2 are both extending acts . Then , M_s is extending if and only if every closed subact N of M_s with $N \cap M_1 = \Theta$ or $N \cap M_2 = \Theta$ is a retract of M_s .

Proof : $\Longrightarrow$)The necessity is clear by proposition(2.5) .

$\Longleftarrow$) Suppose that every closed subact N of M_s with $N \cap M_1 = \Theta$ or $N \cap M_2 = \Theta$ is a retract of M_s. Let A be a closed subact of M_s . Then , there exists a complement B in A such that $A\cap M_2$ is $\cap$-large in B and since A is closed of M_s , so B is closed of M_s by lemma(2.4) . Since $(A\cap M_2) \cap M_1$ is $\cap$-large in $B\cap M_1$ whence M_1 is $\cap$-large in M_1 . Thus, $B\cap M_1 = \Theta$ (as $A \cap (M_2\cap M_1) = A\cap\Theta = \Theta$ which implies that Θ is $\cap$-large in Θ . Then, by hypothesis , $M = B \dot{\cup} B^{/}$ for some $B^{/}$ of M_s and B is a retract of M_s . Now , $A = A\cap M_s = A \cap (B\dot{\cup} B^{/}) = B \dot{\cup}(A \cap B^{/})$. Thus , $A\cap B^{/}$ is closed in M_s (since $A\cap B^{/}$ is closed in A) . Also , $(A\cap B^{/}) \cap M_2 = \Theta$, so by hypothesis $A\cap B^{/}$ is a retract of M_s and hence of $B^{/}$ (since $A\cap B^{/} \subseteq B^{/}$) . Thus, $B^{/} = (A\cap B^{/}) \dot{\cup} N$, where N is subact of $B^{/}$. Now, $M_s = B \dot{\cup} B^{/} = B \dot{\cup} ((A \cap B^{/}) \dot{\cup} N) = (B\dot{\cup} (A\cap B^{/})) \dot{\cup} N = A\dot{\cup}N$. It follows that A is a retract of M_s . Thus , M_s is extending act . ■

Proposition (2.12): Let M_1 and M_2 be S-acts and let $M_s = M_1 \dot{\cup} M_2$. Then , M_1 is M_2-injective if for every subact N of M_s such that $N \cap M_1 = \Theta$, there exists a subact $M^{/}$ of M_s such that $M_s = M_1 \dot{\cup} M^{/}$ and $N \subseteq M^{/}$.

Proof : Suppose that M_1 is M_2-injective . Let $\pi_i : M_s \longrightarrow M_i$, where i = 1,2 be the projection map and let N be a subact of M_s such that $N \cap M_1 = \Theta$. Let $\alpha = \pi_{1|N}$, $\beta = \pi_{2|N}$. By hypothesis there exists an S-homomorphism $f : M_2 \longrightarrow M_1$ such that $f\circ \beta = \alpha$. Denote $M^{/} = \{(f(x), x) \mid x \in M_2 \}$. It is easy to check that $M^{/}$ is a subact of M_s and $M_s = M_1 \dot{\cup} M^{/}$ with $N \subseteq M^{/}$. ■

The following proposition give a condition under which direct sum of extending acts is extending :

Proposition (2.13): Let $M_s = \dot{\cup}_{i=1}^{n} M_i$ be a finite direct sum of relatively injective acts M_i . Then M_s is extending if and only if all M_i are extending .

Proof : $\Longrightarrow$) The necessity is clear (since retract of extending is extending) .

$\Longleftarrow$) Suppose that all M_i are extending and each M_i are relatively injective acts . By induction on n , it is sufficient to prove that M_s is extending when n = 2 . Let $A \subseteq M_s$ be a closed and $A\cap M_1 = \Theta$. By proposition (2.12) , there exists a subact $M^{/}$of M_s such that $M_s = M_1\dot{\cup}M^{/}$ and $A \subseteq M^{/}$. Clearly , $M^{/} \cong M_2$ and hence $M^{/}$is extending . It is obvious that A is closed subact of $M^{/}$. Since $M^{/}$is extending , so by proposition(2.5) A is a

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retract of $M^/$ and then it is a retract of M_s whence $M^/$is a retract of M_s . Similarly any closed $B \subseteq M_s$ with $B \cap M_2 = \Theta$ is a retract of M_s . Thus , by proposition(2.11) , M_s is extending act . ■

3- Principally Extending acts :

Definition (3.1): An S-act M_s is called principally extending act (for short P-extending) if every cyclic subact of M_s is ⋂-large in a retract of M_s . Or equivalently, every cyclic-closed subact of M_s is a retract of M_s .

Definition (3.2): A subact N of S-act M_s is called ⋂-large cyclic (closed) subact if a (closed) subact is contain ⋂-large cyclic subact . This means, there exists $n \in N$ such that nS is ⋂-large subact of N .

An S-act M_s is called ⋂-large cyclic extending act (for short ⋂-large CCS act) if every ⋂-large cyclic-closed subactof M_s is a retract .

From above definition , we note that the property of ⋂-large CCS lies strictly between CS and P-extending properties .

Lemma(3.3): Every retract of ⋂-large cyclic subact of an S-act M_s is ⋂-large cyclic subact .

Proof: Let N be ⋂-large cyclic subact of S-act M_s and N_1 be a retract of N . As N is ⋂-large cyclic subact of M_s , so by definition(3.1) , there exists $n \in N$ such that nS is ⋂-large subact of N . Now, $N = N_1 \dot{\cup} N_2$ (since N_1 is retract of N) . Then, if $n \in N$, then $n \in N_1$ or $n \in N_2$. For $n \in N_1$ which implies that $n = n_1$ and it is easy to check that n_1S is ⋂-large subact of N_1(similarly , if $n \in N_2$) . Thus $N_1(N_2)$ is ⋂-large cyclic subact of N . ■

Corollary(3.4): Every retract of ⋂-large cyclic –closed subact of S-act M_s is ⋂-large cyclic-closed.■

Proposition (3.5): Let $M_s = M_1 \dot{\cup} M_2$ and $N \cap M_1$ be an ⋂-large cyclic subact of M_s , for every ⋂-large cyclic closed subact N of M_s . Then , M_s is P-extending if and only if every ⋂-large cyclic-closed subact N with $N \cap M_1 = \Theta$ or $N \cap M_2 = \Theta$ is a retract .

Proof: The necessary condition is obvious . For the sufficient condition , let nS be ⋂-large subact of N and N be ⋂-large cyclic –closed subact of M_s . If $N \cap M_1 = \Theta$, then we are done . Otherwise $N \cap M_1$ is ⋂-large cyclic subact of M_s , by assumption . Then , let N_1 be maximal ⋂-large extension of $N \cap M_1$ in N , then N_1 is ⋂-large cyclic-closed subact of N with $N_1 \cap M_2 = \Theta$. Hence , by the assumption , N_1 is a retract of M_s . Now, $M_s = N_1 \dot{\cup} N_2$, then $N = N_1 \dot{\cup} (N \cap N_2)$. By corollary(3.4) , $N \cap N_2$ is ⋂-large cyclic-closed subact of M_s with $(N \cap N_2) \cap M_1 = \Theta$ and therefore $N \cap N_2$ is a retract of M_s . Thus , N is a retract of M_s and therefore M_s is P-extending act .■

Proposition (3.6): Let M_s be ⋂-large CCS-act and N subact of M_s . Assume that M_s contains a cyclic ⋂-large subact . Then N is a retract if and only if N is ⋂-large cyclic closed .

Proof : Let $A = xS$ for some $\Theta \neq x \in M_s$ such that A is ⋂-large in M_s . If N is ⋂-large cyclic closed , then by hypothesis N is a retract . Conversely, assume that N is a retract of M_s , then $M_s = N \dot{\cup} N_1$ for some subact N_1 of M_s . Let $\pi : M_s \longrightarrow N$ be projection homomorphism , then , we have $A \cap N = xS \cap N \subseteq \pi(A) = \pi(x)S \subseteq N$ and $\pi(x)S$ is ⋂-large in N . Hence N is ⋂-large cyclic closed subact . ■

Proposition (3.7): Let $M_s = M_1 \dot{\cup} M_2$, where M_1 is a semisimple act . Then , M_s is P-extending if and only if every ⋂-large cyclic-closed subact N with $N \cap M_1 = \Theta$ is a retract .

Proof: The necessary condition is obvious . For the sufficient condition , let N be an ⋂-large cyclic-closed subact of M_s . If $N \cap M_1 = \Theta$, then the proof is complete . On the other hand , since M_1 is semisimple , so we

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get $N \cap M_1$ is a retract of M_1 and so $N=(N \cap M_1)\dot{\cup} N_1$, so by corollary(3.4) N_1 is $\cap$-large cyclic-closed subact of M_s . Since M_s is P-extending act and N_1 is $\cap$-large cyclic-closed subact of M_s with $N_1 \cap M_1 = \Theta$, then by proposition(3.5) N_1 is a retract of M_s . Therefore N is a retract of M_s .

Lemma(3.8):

1- Every fully invariant subact of P-extending act is P-extending .

2- Retract of P-extending act is P-extending .

Proof : 1- The proof is similar to proposition(2.10) by replacing extending act by P-extending and subact by cyclic subact . ■

2-Let N be a retract of P-extending act M_s . Let A be cyclic closed subact of N , then by lemma(2.4) , A is cyclic closed subact of M_s . Since M_s is P-extending , so A is retract of M_s and then of N . Thus , N is P-extending act . ■

4- Relation among Extending S-acts with other Classes of Injectivity :

Proposition(4.1): Every quasi injective act with $\psi_M = i$ is extending .

Proof : Let M_s be any quasi injective act and N be any subact of M_s . Let H be complement of N in M_s . Then , by proposition (2.6) , $N \dot{\cup} H$ is $\cap$-large in M_s . Then , by taking injective envelope , we have $E(M_s) = E(N) \dot{\cup} E(H)$. By theorem(2.4) in [7] , $M_s = (M_s \cap E(N)) \dot{\cup} (M_s \cap E(H))$ and N is $\cap$-large in a retract $M_s \cap E(N)$ of M_s . Thus M_s is extending act . ■

Proposition(4.2): Any quasi injective act M_s with $\psi_M = i$ satisfies the following two conditions :

C_1- condition : Every subact of M_s is $\cap$-large in a retract M_s .

C_2-condition : If a subact A of M_s is isomorphic to a retract of M_s , then A is a retract of M_s .

Proof : Let M_s be any quasi injective S-act and N be a subact of M_s . Consider a complement L of N in M_s , then $N \dot{\cup} L$ is $\cap$-large in M_s . Taking injective envelope , we have $E(M_s) = E(N)\dot{\cup}E(L)$. Thus , by theorem(2.4) in [7] , $M_s = (M_s \cap E(N)) \dot{\cup} (M_s \cap E(L))$ and then N is $\cap$-large in a retract $M_s \cap E(N)$ which is a retract of M_s . Therefore M_s is extending (this means C_1-holds) . Let f: $M^{/} \longrightarrow M_s$ be a monomorphism with $M^{/}$ is a retract of M_s . Since M_s is M-injective , so $M^{/}$ is M-injective . Thus , f is splits and M_s is a retract . ■

From proposition(4.1) , we have every quasi injective act with $\psi_M = i$ is extending act . The following proposition give a condition under which the converse is true , but we need the following definition :

Definition(4.3): An S-act M_s is called a DRI- act if any two subacts of M_s are relatively injective , whenever they form a direct decomposition of M_s (this means that M_i is M_j-injective ($i \neq j = 1,2$) whenever $M_s = M_1 \dot{\cup} M_2$) .

Proposition(4.4): An S-act M_s with $\psi_M = i$ is quasi (continuous) act if and only if M_s is a DRI-CS(extending)-act .

Proof : $\Longrightarrow$) By proposition(4.2) .

$\Leftarrow$) As M_s is extending act , so C_1 -condition is satisfy . Let $A\cong B$ where A and B be subacts of M_s and B be a retract of M_s . Then , $M_s = B \dot{\cup} H$, where H is a subact of M_s . Since M_s is DRI-act , so B is H-injective . Then , A is H-injective whence $A\cong B$. Now , for any monomorphism $f : A\longrightarrow H$ splits , which implies that A is closed (retract) of H . Since H is closed in M_s , thus A is closed in M_s by lemm(2.4) . As M_s is extending act , so A is retract of M_s and C_2-condition is satisfy . Therefore M_s is quasi(continuous) act . ■

It is well-known that every CS-act is P-extending , but the converse is not true in general , the following proposition give under which the converse is true , but we need the following :

Definition(4.5):[1]An S-act M_s is principally self-generator if every $x \in M_s$, there is an S-homomorphism $f : M_s \to xS$ such that $x = f(x_1)$ for $x_1\in M_s$.

Proposition(4.6): Let M_s be principal and principal self-generator act . Then , the following statements are equivalent :

1- M_s is CS-act ,

2- M_s is P-extending ,

3- M_s is $\cap$-reversible .

Proof : (1 $\longrightarrow$2) It is obvious .

(2 $\longrightarrow$3) Let N be a non-zero subact of M_s. Since M_s is principal self-generator , so there exists S-epimorphism $\alpha : M_s \longrightarrow N$ and since M_s is principal , so N is principal . As M_s is P-extending , so N is $\cap$-large in a retract of M_s . But Θ_s and M_s are only the retract of M_s (since M_s is principal and every principal is indecomposable) . Thus N is $\cap$-large subact of M_s . Hence M_s is $\cap$-reversible act .

(3 $\longrightarrow$1) Let N be a subact of M_s . As M_s is $\cap$-reversible act , so N is $\cap$-large subact of M_s . Since M_s is principal so it is indecomposable ([6], p.66) , then the only retract of M_s are Θ_s and M_s . Thus N is $\cap$-large in a retract of M_s . ■

Lemma(4.7):[1] Every non-zero subact N of centered S-act M_s over semigroup with zero has maximal intersection large in M_s called closure of N in M_s .

Proposition(4.8): Let M_s be anon-zero principal and principal self-generator act . Then , the following statements are equivalent :

1- M_s is CS-act ,

2- M_s is $\cap$-large CCS-act ,

3- M_s is P-extending act ,

4- M_s is $\cap$-reversible act .

Proof : (1 $\longrightarrow$2) It is obvious .

(2 $\longrightarrow$3) Let $mS(\neq \Theta)$ be any cyclic subact of M_s and N be the closure of mS in M_s by lemma(4.7) . Then , N is $\cap$-large cyclic closed subact of M_s . Thus , by hypothesis N is a retract of M_s and then M_s is P-extending .

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(3 ⟶4) By proposition(4.6) . Now , we prove in another way . Let N be a non-zero subact of M_s and let N_1 be a relative complement of N in M_s , then $N \dot{\cup} N_1$ is ⋂-large in M_s by proposition(2.6) . Thus , N_1 is closed in M_s . Since M_s is principal and principal self-generator , so N_1 is principal and closed . By hypothesis N_1 is a retract of M_s . As M_s is principal so it is indecomposable ([6], p66) , so the only retract of M_s are Θ_s and M_s and since $N_1 \neq M_s$, so $N_1 = \Theta$. Thus, N is ⋂-large in M_s and M_s is ⋂-reversible .

(4 ⟶1) Let N be a subact of M_s . Since M_s is ⋂-reversible , so N is ⋂-large in M_s . As M_s is principal so it is indecomposable ([6], p66) , so the only retract of M_s are Θ_s and M_s . Thus , N is ⋂-large in a retract of M_s and M_s is CS- act . ■

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