Multiplicative Noise Reduction Model Based on Logarithmic Regularization

Ogada Achieng Elisha∗1, Onyango Lawrence Omondi †2, Ondiek David Manyanga‡3, and Ongowe Fredrick Odhiambo §4

1,2,3Department of Mathematics, Egerton University, P. O. Box 536-20115, Egerton, Kenya
4School of Mathematics, CBPS College, University of Nairobi, P.O. Box 30197-00100. Nairobi, Kenya

Abstract

In this paper, we have proposed a formulation for alleviating multiplicative (speckle) noise. The formulation developed is quite different from the traditional total variation (TV), which are basically the TV functional only. We have presented a logarithm-based functional that is a function of the absolute of the gradient of the image function \( u \). Our functional has been inspired by works in the area of fluid dynamics, in the flow of Prandtl-Erying type of fluids. Logarithm-based functions grow rather slowly, and are, therefore, believed to be fairly gentle on image features in the real work of image restoration (image denoising). For the proposed functional we have proved the existence and uniqueness of the corresponding minimization problem. To prove the existence and uniqueness of the solution of the evolution problem, we have reexpressed the corresponding evolution problem as a semi-discrete minimization energy functional. From the minimization problem, we have obtained a discrete evolution problem, for which we have proved the existence and uniqueness of the solution. We have shown that the obtained sequence of approximate solutions actually converges to the unique minimizer of the variational problem. Numerical implementation is included in the last section to show the results due the proposed formulation.

Keywords: Total Variation, Strictly convex, Image denoising, Edge indicator, speckle noise, Bounded Variation.

1 Introduction

In most real life applications, speckle noise is more prevalent than the standard Gaussian additive noise. This type of noise is generally referred to as multiplicative noise. It occurs when a coherent wave sent from the radar is reflected on a courser surface, compared to the amplitude of the radar. The wave is then corrupted by a noise of larger amplitude; thereby causing speckled effects on the image [1, 2]. Encountered in such imaging applications as ultrasound imaging, laser imaging, synthetic aperture radar(SAR), microscope images and sonar(SAS), among others, multiplicative noise has been the subject widespread study in image processing [3, 4, 5, 6]. Multiplicative noise usually leads to observed images with characteristic granular appearance, due to very low signal to noise ratio (SNR) [3, 7].

In contrast to the traditional Gaussian additive noise, where noise is added, applications involving speckle noise have the noise element multiplied to the ideal image [3, 8, 9], and in most cases the noise is described by probability density functions that are non-Gaussian. The most common probability densities being Poisson, Rayleigh and Gamma distributions [1, 10].

∗Email: achiengelisha@gmail.com; (Corresponding Author)
†Email: lawioniyi@gmail.com
‡Email: davidondiek@yahoo.com
§Email: fredongove@gmail.com
The denoising of an image degraded by a multiplicative noise, therefore, is basically the estimation of reflectance of the underlying image scene [3]. The inverse nature of this problem calls for regularization, where it is assumed that the underlying image is piece-wise smooth. Toward this end, many approaches have been proposed; namely, Bayesian framework, using Markov random field priors [11], variational techniques [9, 12, 13], wavelet methods [14], sparse analysis method [15], receding horizon control [16], among others.

The variational methods, for instance, lead to optimization problems which result in partial differential equations (PDE’s), from which the solution is computed. They generally extend the application of spatial filters to obtain piece-wise smooth estimates of the ideal image, in a manner that is sensitive to the topography of the underlying image structure.

As opposed to the Gaussian additive noise, which has a quadratic data term, the multiplicative denoising variational formulation usually has log-likelihood fidelity term, which is non-quadratic [see 8, 3, 1, 15]. Various non-smooth regularization terms have been proposed, and TV regularizer and its variants have been the subject of large body of literature [see 9, 8, and references therein].

Let \( \Omega \subset \mathbb{R}^2 \) be open and bounded. If \( f : \Omega \to \mathbb{R} \), then multiplicative noise model generally assumes the form:

\[
f = u \eta,\tag{1.1}
\]

where \( u \) is the ideal image, \( f \) is the observed noisy image and \( \eta \) is the noise.

In many applications, the mean and variance of noise are \textit{apriori} assumed known to satisfy the following conditions

\[
\frac{1}{|\Omega|} \int_{\Omega} \eta \, dx = 1 \tag{1.2}
\]

and

\[
\frac{1}{|\Omega|} \int_{\Omega} (\eta - 1)^2 \, dx = \sigma^2. \tag{1.3}
\]

Most of the available models for multiplicative noise rely on the logarithm transform, where the noise model is essentially converted to the additive model using logarithms [8]; thus from (1.1) we have

\[
\log f = \log u + \log \eta. \tag{1.4}
\]

For conditions to be fulfilled, Rudin et al., [9] proposed the following model for a Gaussian white noise

\[
\min_{u \in BV(\Omega)} \left\{ F(u) = \int_{\Omega} \left| \nabla u \right| \, dx + \int_{\Omega} \left( H(u, f) \right) \, dx \right\}, \tag{1.5}
\]

where

\[
H(u, f) = \lambda_1 \int_{\Omega} \frac{f}{u} \, dx + \lambda_2 \int_{\Omega} \left( \frac{f}{u} - 1 \right)^2 \, dx.
\]

The formulation is above is convex only for \( \lambda_1 \geq 2 \lambda_2 \). This means that the method is hardly entirely convex if the parameters \( \lambda_1 \) and \( \lambda_2 \) are obtained dynamically during the evolution process. However, if the parameters are fixed, the minimization problem results into a sequence of constant function \( u \to \infty \) [see 17, 9].

Taking \textit{speckle} noise with mean equal to one, and assuming the following Gamma distribution for the noise

\[
g(\eta) = \frac{L^L}{L!} \eta^{L-1} \exp(L\eta)_{\eta \geq 0}. \tag{1.6}
\]

Aubert et al. [1] have derived multiplicative noise model that is convex only for \( \eta \in [0, 2f] \), based on the maximum \textit{aposteriori} (MAP) on \( P(u/f) \), where \( P(u) \) followed Gibb’s prior, and the likelihood component modeled as a Gamma distribution. The model is given as

\[
 u = \arg \min_{u \in S(\Omega)} \left\{ \int_{\Omega} |Du| + \lambda \int_{\Omega} H(u, f) \, dx \right\}, \quad H(s, f) = \log s + \frac{f}{s}, \tag{1.7}
\]
where $S(\Omega) = \{ u > 0, u \in BV(\Omega) \}$. A presentation of proof of the existence and uniqueness of solution of the minimization problem, and indeed that of the corresponding evolution system has been offered by the authors.

For other approaches we refer readers to the works of Huang et al., [18, and references there in], who have also used MAP criterion to derive a formulation based on an exponential transform under the assumption of a Gamma distribution.

It is rather obvious that in all these reviews, much as various fidelity terms have been formulated, the regularization kernel has predominantly been that of the original TV-regularization. In this paper, however, we present a logarithm-based minimization problem. Energy problems with logarithmic growth condition appeals more than just for mathematical taste. Logarithmic regularization with respect to variational problems with linear growth in first order weak partial derivatives has been studied in solid and fluid mechanics.

Perfect plasticity has been modeled by Fuchs et al., [19] in respect of plasticity with logarithmic hardening. Additionally, in [20], the authors have presented a discussion on local minimizers of variational problems which occur in the theory fluids of Prandtl-Erying type and plastic materials with logarithmic hardening. On the basis of the foregoing, we present a technique based on logarithmic regularization in multiplicative (speckle) noise reduction.

This work also aims to extend the work of Dong et al., [8]. We, however, propose new multiplicative noise reduction model where the regularization kernel has logarithmic growth component. In this model, the diffusion coefficient is a logarithm based function of the local image gradient. We then prove the existence, uniqueness of the solution of the energy functional. Analysis of the corresponding evolution problem, in terms of the existence and uniqueness of the solution, is also presented. Finally, we give experimental results to illustrate the effectiveness of the model in speckle noise reduction.

2 Proposed Model for Multiplicative Noise Reduction

In this section, motivated by the works by [21, 8, 22, 23], we propose a new logarithm-based variational model for speckle noise reduction. The model takes the form

$$u = \arg \min_{u \in BV(\Omega)} \left\{ \int_{\Omega} \Phi(|Du|) + \lambda \int_{\Omega} \left[ u + f \log \left( \frac{1}{u} \right) \right] dx \right\},$$

(2.1)

where $\Phi(s) = s \left[ \log(1+s) + \frac{1}{s+1} \right]$, $\lambda$ is the fidelity parameter, and $f > 0, u > 0$.

And right from the outset we remark from equation (2.1) that the model above satisfies the following desirable properties,

(H1) Considering the regularization part of the minimization problem see that

$$\Phi''(s) = \left[ \frac{1}{1+s} + \frac{1}{(1+s)^2} + \frac{2}{(1+s)^3} \right] > 0.$$

Hence $\Phi(s)$ is strictly convex; since $s > 0$.

(H2) In the fidelity term, let $H(z) = z + f \log \left( \frac{1}{z} \right)$, then

$$H''(z) = \frac{f}{z^2} > 0.$$

Hence $H(z)$ is strictly convex

(H3) $\Phi(0) = 0, \lim_{s \to \infty} \Phi(s) = +\infty$; implying that $\Phi(s)$ is a coercive non-decreasing function $\forall s \in \mathbb{R}^+$.
(H4) \[
\lim_{s \to \infty} \Phi''(s) = \lim_{s \to \infty} \frac{\Phi'(s)}{s} = 0,
\]
and
\[
\lim_{s \to \infty} \frac{\Phi''(s)}{\Phi'(s)/s} = 0.
\]
This indicates a good potentiality for preserving edges, since the tangential and normal diffusions, in respect of the image isophote lines, proceed towards zero at different rates with the normal diffusion diminishing at a faster rate.

(H5) The flux function \( s\Phi'(s) = \left[\log(1 + s) + \frac{s}{1+s} + \frac{2s^2}{(1+s)^2}\right] \) is a non-decreasing function for all \( s \in \mathbb{R}^+ \).

This feature may also be observed from Figure 1 below:

![Figure 1: A plot of the Flux Function \( s\Phi'(s) \)](image)

From the above properties it is easy to see that the existence and uniqueness of the minimization problem (2.1), can be proved in some suitable space; especially the space of functions of Bounded Variation (\( BV(\Omega) \)-space).

**Lemma 2.1.** [see 24, p. 64]: Let \( \Phi : \mathbb{R} \to \mathbb{R}^+ \) be convex, even, nondecreasing on \( \mathbb{R}^+ \) with linear growth at infinity. Also let \( \Phi^\infty \) be the recession function of \( \Phi \) defined by

\[
\Phi^\infty(\omega) = \lim_{s \to \infty} \frac{\Phi(s\omega)}{s}.
\]

Then for \( u \in BV(\Omega) \) and setting \( \Phi(\theta) = \Phi(|\theta|) \) we have,

\[
\int_\Omega \Phi(Du) = \int_\Omega \Phi(|\nabla u|)dx + \Phi^\infty(1) \int_\Omega D^s u.
\]

This implies that \( u \to \int_\Omega \Phi(Du) \) is lower semi-continuous for the \( BV(\Omega) \) – weak* topology.

**Lemma 2.2** (Fatou’s Lemma). [see 25]: Let \( f_n : \Omega \to [0, +\infty] \) be a measurable function. Then

\[
\int_\Omega \liminf_{n \to \infty} f_n dx \leq \liminf_{n \to \infty} \int_\Omega f_n dx \tag{2.2}
\]
Lemma 2.3. [see 24, p.267] Let \( u \in BV(\Omega) \), and \( \Phi \) be a convex, nondecreasing function on \( \mathbb{R}^+ \) with linear growth properties, and \( \psi_{a,b} \) be the cut-off function defined by

\[
\psi_{a,b}(x) = \begin{cases} 
a & \text{if } x \leq a \\
x & \text{if } a \leq x \leq b \\
b & \text{if } x \geq b. 
\end{cases}
\]  \quad (2.3)

Then

\[
\int_{\Omega} \Phi(D\psi_{a,b}(u)) \leq \int_{\Omega} \Phi(Du) \quad (2.4)
\]

2.1 Existence and Uniqueness for the Minimization Problem

By the following theorem we prove the existence and uniqueness of the solution of the minimization problem (2.1).

Theorem 2.1. Let \( f \) be in \( L^\infty(\Omega) \) such that \( \inf_{x \in \Omega} f > 0 \), then the minimization problem (2.1) has a unique solution \( u \in BV(\Omega) \) such that

\[
0 < \inf_{x \in \Omega} f \leq u \leq \sup_{x \in \Omega} f. \quad (2.5)
\]

Proof. From (2.1), let us write

\[
F(u) = \int_{\Omega} \Phi(|Du|) + \lambda \int_{\Omega} \left[ u + f \log \left( \frac{1}{u} \right) \right] dx. \quad (2.6)
\]

Observe that \( F(u) \) is an increasing function on \( \mathbb{R}^+ \), and that the data fidelity term reaches a minimum at \( u = f \). This implies

\[
H(f) = f + f \log \left( \frac{1}{f} \right), \quad (2.7)
\]

and hence

\[
F(u) \geq \int_{\Omega} \left[ f + f \log \left( \frac{1}{f} \right) \right] dx, \quad \text{for } u \in \{ u > 0, BV(\Omega) \} \quad (2.8)
\]

This shows that \( F(u) \) bounded below \( \forall 0 < u \in BV(\Omega) \). And given the coercivity of \( F(u) \) it implies that there exits a minimizing sequence \( \{ u_n \} \subset BV(\Omega) \) for the minimization problem (2.1).

Furthermore, denote \( a = \inf_{x \in \Omega} f \) and \( b = \sup_{x \in \Omega} f \), and note that the data fidelity function \( H(z) \) is decreasing for \( z \in (0, f) \) and increasing for \( z \in (f, +\infty) \). Then, if \( C \geq f \), it will always be that

\[
\min(z, C) + f \log \left( \frac{1}{\min(z, C)} \right) \leq z + f \log \left( \frac{1}{z} \right). \quad (2.9)
\]

Thus if \( C = b = \sup_{x \in \Omega} f \), then it implies

\[
\int_{\Omega} \left[ \min(u, C) + f \log \left( \frac{1}{\min(u, C)} \right) \right] dx \leq \int_{\Omega} \left[ u + f \log \left( \frac{1}{u} \right) \right] dx. \quad (2.10)
\]

And on the strength of Lemma 2.3 we may have

\[
\int_{\Omega} \Phi(D\min(u, b))) \leq \int_{\Omega} \Phi(Du). \quad (2.11)
\]

Thus combining (2.10) and (2.11) we conclude that

\[
F(\min(u, b)) \leq F(u). \quad (2.12)
\]
Conversely, using a similar process we may obtain that
\[ F(\sup(u, a)) \leq F(u), \quad \text{where } a = \inf f. \tag{2.13} \]
Hence, from above, we deduce that \( a \leq u_n \leq b \), and thus it implies \( u_n \) is bounded in \( L^1(\Omega) \). We then show that there exist \( u \in BV(\Omega) \) such that
\[ F(u) = \min_{v \in BV(\Omega)} F(v) \tag{2.14} \]
From the boundedness of \( u_n \), as shown above, and given that \( H(z) \in C[a, b] \), it implies \( H(u_n) \) is also bounded. Additionally, observe that \( \{u_n\} \subset BV(\Omega) \). Hence there exist a constant \( M \) such that
\[ \int_\Omega \Phi(Du_n) + \int_\Omega H(u_n) dx \leq M \tag{2.15} \]
This implies
\[ \int_\Omega \Phi(Du_n) \leq M. \tag{2.16} \]
Hence, there exists a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) and a function \( u \in BV(\Omega) \) such that
\[ u_{n_k} \rightarrow u, \quad \text{strongly in } L^1(\Omega), \]
\[ u_{n_k} \rightharpoonup u, \quad \text{weak}^* \text{ in } BV(\Omega). \]
And by Lemma 2.1 and Lemma 2.2, we have that
\[ F(u) \leq \liminf_{k \to \infty} F(u_{n_k}) = \min_{v \in BV(\Omega)} F(v). \]
Hence, there exists a solution of the minimization problem (2.1). And uniqueness of the solution follows from the strict convexity of the functional \( F(u) \).

\[ \square \]

2.2 The Associated Evolution Problem

From the minimization problem (2.1) we derive the following Euler-Lagrange system in the distributional sense
\[ 0 = -\text{div} (C(|\nabla u|) \nabla u) + \lambda H'(u) \quad x \in \Omega, \tag{2.17} \]
\[ \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega, \tag{2.18} \]
where
\[ C(s) = \begin{cases} \log(1 + s) + \frac{1 + s}{1 + s} + \frac{2 + s}{(1 + s)^2}, & s \neq 0; \\ 4, & s = 0, \end{cases} \tag{2.19} \]
\[ H'(z) = \left(1 - \frac{f}{z}\right). \tag{2.20} \]

3 Semi-Discrete Form of Evolution Equation

In this section, motivated by works of [24, 13], we seek consider the problem (2.17)–(2.20) in a semi-discrete context. That is, instead of immersing the problem into the traditional continuous dynamical system, we embed (2.17)–(2.20) into a discrete time dynamical system. However, before that consider the following:
Given that $BV(\Omega) \subset L^2(\Omega)$, and letting $\mathcal{F}(u) = \int \Phi(|\nabla u|)dx$, we seek to extend $\mathcal{F}$ over $L^2(\Omega)$ under the same notation, and have

$$\mathcal{F}(u) = \begin{cases} 
\int_\Omega \Phi(|\nabla u|)dx, & \text{if } u \in BV(\Omega), \\
+\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega),
\end{cases} \quad (3.1)$$

where $\Phi(s) = s \left[ \log(1 + s) + \frac{s}{1 + s} \right]$.

With the sub-differential $\partial \mathcal{F}(u)$ of $\mathcal{F}(u)$ above being defined as

$$\partial \mathcal{F}(u) = \{ v \in L^2(\Omega) : \langle v, w - u \rangle_{L^2(\Omega)}, \forall w \in BV(\Omega) \cap L^2(\Omega) \}, \quad (3.2)$$

it is easy to find that in this particular case the sub-differential, $\partial \mathcal{F}(u)$, has a singleton element $\{ v \}$ given by

$$v = -\text{div} (C(|\nabla u|)\nabla u), \quad (3.3)$$

where $C(s)$ is as defined in (2.19) above.

Without loss of generality we shall let $\lambda = 1$, we now embed the equation (2.17) in to a discrete time dynamical system, where $t_{m+1} - t_m = \Delta t \in \mathbb{R}^+$, $t_0$ is assumed known apriori, and we define $u_m = u(\cdot, t_m)$.

Hence we have

$$0 \in \frac{u_{m+1} - u_m}{\Delta t} + \partial \mathcal{F}(u_{m+1}) + H'(u_{m+1}), \text{ in } \Omega_T, \quad (3.4)$$

where $\Omega_T = \Omega \times (0, T)$.

From (3.9) we wish to consider the discrete energy functional

$$\inf_{u \in BV(\Omega), u \geq 0} \left\{ E(u, u_m) = \int_\Omega \left( \frac{u^2}{2} - u_m u + \Delta t H(u) \right) dx + \Delta t \mathcal{F}(u) \right\}, \quad (3.5)$$

Given that $BV(\Omega) \subset L^2(\Omega)$, and letting $\mathcal{F}(u) = \int \Phi(|\nabla u|)dx$, we seek to extend $\mathcal{F}$ over $L^2(\Omega)$ under the same notation, and have

$$\mathcal{F}(u) = \begin{cases} 
\int_\Omega \Phi(|\nabla u|)dx, & \text{if } u \in BV(\Omega), \\
+\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega),
\end{cases} \quad (3.6)$$

where $\Phi(s) = s \left[ \log(1 + s) + \frac{s}{1 + s} \right]$.

With the sub-differential $\partial \mathcal{F}(u)$ of $\mathcal{F}(u)$ above being defined as

$$\partial \mathcal{F}(u) = \{ v \in L^2(\Omega) : \langle v, w - u \rangle_{L^2(\Omega)}, \forall w \in BV(\Omega) \cap L^2(\Omega) \}, \quad (3.7)$$

it is easy to find that in this particular case the sub-differential, $\partial \mathcal{F}(u)$, has a singleton element $\{ v \}$ given by

$$v = -\text{div} (C(|\nabla u|)\nabla u), \quad (3.8)$$

where $C(s)$ is as defined in (2.19) above. Without loss of generality we shall let $\lambda = 1$, we now embed the equation (2.17) in to a discrete time dynamical system, where $t_{m+1} - t_m = \Delta t \in \mathbb{R}^+$, $t_0$ is assumed known apriori, and we define $u_m = u(\cdot, t_m)$. Hence we have

$$0 \in \frac{u_{m+1} - u_m}{\Delta t} + \partial \mathcal{F}(u_{m+1}) + H'(u_{m+1}), \text{ in } \Omega_T, \quad (3.9)$$

where $\Omega_T = \Omega \times (0, T)$.

From (3.9) we wish to consider the discrete energy functional

$$\inf_{u \in BV(\Omega), u \geq 0} \left\{ E(u, u_m) = \int_\Omega \left( \frac{u^2}{2} - u_m u + \Delta t H(u) \right) dx + \Delta t \mathcal{F}(u) \right\}. \quad (3.10)$$
4 Existence and Uniqueness of the Solution Sequence for the Discrete Minimization Problem

In this section we present the proof of the existence and uniqueness of the solution of the semidiscrete minimization problem (3.10).

**Theorem 4.1.** Let \( u_0 \in BV(\Omega) \cap L^\infty(\Omega) \) be given, where \( \inf_\Omega u_0 > 0 \). Then there exists a unique sequence \( \{u_m\} \subset BV(\Omega) \) satisfying

\[
\inf \left( \inf_\Omega f, \inf_\Omega u_0 \right) \leq u_m \leq \sup \left( \sup_\Omega f, \sup_\Omega u_0 \right), \quad (4.1)
\]

\( \Delta t > 0 \), such that

\[
\mathcal{F}(u_m) \leq \mathcal{F}(u_0) + \int_\Omega \left[ H(u_0) - \left( f + f \log \left( \frac{1}{f} \right) \right) \right] dx \quad (4.2)
\]

**Proof.** The proof of condition (4.1) can be attained through a process similar to the proof of condition (2.5) in Theorem 2.1 above. However, as for uniqueness, we see that since the potential function of \( \mathcal{F}(u) \) in (3.10) is strictly convex, we only need to show the strict convexity of the component, \( \gamma(u) = \frac{u^2}{2} - u_m u + \Delta t H(u) \). That is, we must have \( \gamma''(u) = \Delta t \left( \frac{d}{dt} \right) + 1 > 0 \). Visibly, this is possible if and if only \( \Delta t \geq 0 \). However, since under the discrete approximation \( \Delta t \neq 0 \), we find the condition for strict convexity of the problem can only be attained if \( \Delta t > 0 \), since \( f > 0 \).

To show condition (4.2), we recognize that \( E(u_{m+1}, u_m) \leq E(u_m, u_m) \). Hence we have:

\[
\int_\Omega \left[ \frac{u_{m+1}^2}{2} - u_m u_{m+1} + \Delta t H(u_{m+1}) \right] dx + \Delta t \mathcal{F}(u_{m+1}) \leq \int_\Omega \left[ \frac{u_m^2}{2} - u_m^2 + \Delta t H(u_m) \right] dx + \Delta t \mathcal{F}(u_m)
\]

Rearranging the above inequality we obtain

\[
\frac{1}{2} \int_\Omega (u_{m+1} - u_m)^2 dx + \Delta t \int_\Omega (H(u_{m+1}) - H(u_m)) dx + \Delta t (\mathcal{F}(u_{m+1}) - \mathcal{F}(u_m)) \leq 0 \quad (4.4)
\]

Summing (4.4) from \( m = 0 \) to \( m = M - 1 \) we obtain the inequality

\[
\frac{1}{2} \sum_{m=0}^{M-1} \int_\Omega (u_{m+1} - u_m)^2 dx \leq \Delta t \int_\Omega H(u_0) dx - \Delta t \int_\Omega H(u_M) dx + \Delta t \mathcal{F}(u_0) - \Delta t \mathcal{F}(u_M)
\]

\[
\leq \Delta t \int_\Omega H(u_0) dx - \Delta t \int_\Omega H(u_M) dx + \Delta t \mathcal{F}(u_0), \quad (4.5)
\]

Which by (4.1) becomes

\[
\frac{1}{2} \sum_{m=0}^{M-1} \int_\Omega (u_{m+1} - u_m)^2 dx \leq \Delta t \int_\Omega H(u_0) dx - \Delta t \int_\Omega H(f) dx + \Delta t \mathcal{F}(u_0) < \infty \quad (4.6)
\]

Additionally, it is easy to see from (4.4) that

\[
\Delta t \int_\Omega (H(u_{m+1}) - H(u_m)) dx + \Delta t (\mathcal{F}(u_{m+1}) - \mathcal{F}(u_m)) \leq 0 \quad (4.7)
\]

From which by summation we obtain that

\[
\mathcal{F}(u_{m+1}) \leq \mathcal{F}(u_0) + \int_\Omega H(u_0) dx - \int_\Omega H(u_{m+1}) dx. \quad (4.8)
\]

And since \( H(f) = f + f \log \left( \frac{1}{f} \right) \) is a minimum value of \( H \), we may safely assert that \( H(u_{m+1}) \geq H(f) \). Hence we conclude that

\[
\mathcal{F}(u_{m+1}) \leq \mathcal{F}(u_0) + \int_\Omega H(u_0) dx - \int_\Omega H(f) dx, \quad (4.9)
\]

which confirms (4.2). \( \Box \)
4.1 Euler Lagrange Equation for the Discrete Minimization Problem

The semi-discrete Euler-Lagrange equation associated with the semi-discrete minimization problem (3.10), and also satisfying (3.9), for a sequence \( \{u_m\} \), is given in distributional sense by

\[
0 = \frac{u_{m+1} - u_m}{\Delta t} - \text{div}(C(|\nabla u_{m+1}|)\nabla u_{m+1}) + \left(1 - \frac{f}{u_{m+1}}\right),
\]

(4.10)

with

\[
\frac{\partial u_{m+1}}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\]

It then remains to show the convergence of the sequence \( \{u_m\} \) to the solution \( u \).

4.2 Convergence Of The Solution Sequence to the minimizer \( u \)

Through the statement and proof of following theorem we show that the sequence \( u_m \) of solutions converges to the unique minimizer \( u \) of the problem (2.1).

**Theorem 4.2.** Let \( f, u_0 \in BV(\Omega) \cap L^\infty(\Omega) \) such that \( \inf_{\Omega} f > 0, \inf_{\Omega} u_0 > 0 \) be given, and that \( \Delta t > 0 \). Also define an arbitrary \( v \in BV(\Omega) \cap L^2(\Omega) \). Then there exists \( u \in BV(\Omega) \) such that \( u_m \to u \), where \( u_m \) is as defined in (3.9), and \( u \) solution of

\[
0 \in -\text{div}(C(|\nabla u|)\nabla u) + H'(u)
\]

where \( C(s) \) and \( H'(z) \) is as defined in (2.19) and (2.20), respectively.

**Proof.** From the estimate (4.6) we see that

\[
u_{m+1} \to u \quad \text{strongly in} \quad L^2(\Omega)
\]

(4.11)

Also observe from (4.2) we deduce that up to a subsequence

\[
u_m \to u \quad \text{weakly in} \quad BV(\Omega).
\]

(4.12)

It also implies from (4.1) that

\[
u_m \to u \quad \text{strongly in} \quad L^1(\Omega).
\]

(4.13)

Consider (3.9) with the singleton set sub-differential \( \partial \mathcal{F}(u) \) being given as indicated by (3.8). By deduction we multiply (3.9) through by \( w - u_{m+1} \) and integrating over \( \Omega \) to obtain

\[
\int_{\Omega} \partial \mathcal{F}(u_{m+1})(w - u_{m+1})dx \leq -\int_{\Omega} \left[\frac{u_{m+1} - u_m}{\Delta t} + H'(u_{m+1})\right](w - u_{m+1})dx
\]

(4.14)

Applying convexity to the above inequality we have

\[
\mathcal{F}(w) + H(w) \geq \mathcal{F}(u_{m+1}) - \int_{\Omega} \left[\frac{u_{m+1} - u_m}{\Delta t}\right](w - u_{m+1})dx + H(u_{m+1})
\]

(4.15)

Guided by the above convergences, and the lower continuity of \( \mathcal{F}(u) \) in the context of Lemma 2.1 we deduce that that

\[
F(w) = \mathcal{F}(w) + H(w) \geq \mathcal{F}(u) + H(u) = F(u).
\]

(4.16)

Hence \( u \) is a minimizer, and the strict convexity of the potentials guarantees uniqueness.
5 Numerical Experiments

Here, we present numerical implementation and results from experiments based on the proposed model. We have used the classical Perona-Malik (PM) scheme [26] and the Additive Operator Scheme by Weickert [29].

The experiments in this work were performed on a Compaq610 computer, having Intel(R) Core(TM)2 Duo CPU T5870 @ 2.00GHz, physical RAM of 4.00GB, and Windows 8 Professional, 64-bit Operating System, on MATLAB R2013b.

Image restoration performance was measured in terms of the Peak-Signal-to-Noise Ratio (PSNR), iteration steps and visual effect. The iteration stopping mechanism was based upon the maximal PSNR. PSNR values were obtained according to the formulae by Durand et al. [17], and is given by

$$\text{PSNR}(u, u_0) = 10 \log_{10} \frac{MN}{\max u_0 - \min u_0} \left\| u - u_0 \right\|_2^2 \text{dB}$$

where $u_0$ denotes the noise free image, $u$ is the denoised image, $M \times N$ is the dimension of image, and $\max u_0 - \min u_0$ yields the gray scale range of the original image.

5.1 Numerical Implementation

5.1.1 PM–Scheme

We have proposed a numerical scheme similar to the original PM method, whereby equations (2.17)–(2.18) are discretized as follows

$$C_{N_{i,j}}^n = \frac{\log(1 + |\nabla_N u_{i,j}|)}{|\nabla_N u_{i,j}|} + \frac{1}{1 + |\nabla_N u_{i,j}|} + \frac{2 + |\nabla_N u_{i,j}|}{(1 + |\nabla_N u_{i,j}|)^2}$$

$$C_{S_{i,j}}^n = \frac{\log(1 + |\nabla_S u_{i,j}|)}{|\nabla_S u_{i,j}|} + \frac{1}{1 + |\nabla_S u_{i,j}|} + \frac{2 + |\nabla_S u_{i,j}|}{(1 + |\nabla_S u_{i,j}|)^2}$$

$$C_{E_{i,j}}^n = \frac{\log(1 + |\nabla_E u_{i,j}|)}{|\nabla_E u_{i,j}|} + \frac{1}{1 + |\nabla_E u_{i,j}|} + \frac{2 + |\nabla_E u_{i,j}|}{(1 + |\nabla_E u_{i,j}|)^2}$$

$$C_{W_{i,j}}^n = \frac{\log(1 + |\nabla_W u_{i,j}|)}{|\nabla_W u_{i,j}|} + \frac{1}{1 + |\nabla_W u_{i,j}|} + \frac{2 + |\nabla_W u_{i,j}|}{(1 + |\nabla_W u_{i,j}|)^2}$$

$$\text{div}_{i,j}^n = \left[ C_{N_{i,j}}^n \nabla_N u_{i,j} + C_{S_{i,j}}^n \nabla_S u_{i,j} + C_{E_{i,j}}^n \nabla_E u_{i,j} + C_{W_{i,j}}^n \nabla_W u_{i,j} \right]$$

$\lambda$ is dynamically determined according to the following discretization scheme

$$\lambda^n = \frac{1}{|\Omega|} \sum_{i,j} \text{div}_{i,j}^n (1 - \frac{f_{i,j}}{w_{i,j}}), \quad \text{where } |\Omega| = MN \text{ is the size of image.}$$

Hence from (5.1) and (5.2) we have

$$u_{i,j}^{n+1} = u_{i,j}^n + \tau \text{div}_{i,j}^n - \lambda^n \left( 1 - \frac{f_{i,j}}{w_{i,j}} \right)$$

$$u_{i,j}^0 = f_{i,j}$$

$$u_{i,0}^n = u_{i,1}^n, \quad u_{0,j}^n = u_{1,j}^n, \quad u_{M,j}^n = u_{M-1,j}^n \text{ and } u_{i,N}^n = u_{i,N-1}^n,$$

where

$$\nabla_N u_{i,j} = u_{i-1,j} - u_{i,j}, \quad \nabla_S u_{i,j} = u_{i+1,j} - u_{i,j}$$

$$\nabla_E u_{i,j} = u_{i,j+1} - u_{i,j}, \quad \nabla_W u_{i,j} = u_{i,j-1} - u_{i,j},$$

for $i = 0, 1, 2, \ldots, N$ and $j = 0, 1, 2, \ldots, M$. 

\[\]
5.1.2 AOS–Scheme

Using a scheme similar to the one in [29], the problem (2.17)–(2.18) can be discretized as

\[ \lambda^0 = 0, \]

\[ u^{n+1} = \frac{1}{m} \sum_{i=1}^{m} [I \cdot m A_i(u^k)]^{-1} [u^n + \lambda^n \tau(1 - \frac{f}{u^n})], \]

\[ \text{div}^n = (u^{n+1} - u^n) / \tau, \]

\[ \lambda^n = \frac{1}{|\Omega|} (1 - \frac{f}{u^n}) \text{div}^n, \]

\[ u_{i,j}^0 = f_{i,j} = f(ih, jh), \]

\[ u_{i,j}^n = u_{i,j}^n, \quad u_{M,i}^n = u_{M-1,i}^n, \quad u_{i,N}^n = u_{i,N-1}^n, \]

where \( A_i(u^n) = [a_{i,j}(u^n)] \),

\[ a_{i,j}(u^n) := \begin{cases} \frac{C^n_i + C^n_j}{2h^2}, & [j \in \mathcal{N}(i)], \\ - \sum_{N \in \mathcal{N}(i)} \frac{C^n_i + C^n_N}{2h^2}, & (j = i), \\ 0, & (\text{else}), \end{cases} \]

and

\[ C^n_i := \left[ \frac{\log(1 + |\nabla u_{i,j}^n|)}{|\nabla u_{i,j}^n|} + \frac{1}{1 + |\nabla u_{i,j}^n|} + \frac{2 + |\nabla u_{i,j}^n|}{(1 + |\nabla u_{i,j}^n|)^2} \right], \]

where

\[ |\nabla u_{i,j}^n| = \frac{1}{2} \sum_{p,q \in \mathcal{N}(i)} \frac{|u^n_p - u^n_q|}{2h}, \]

where \( \mathcal{N}(i) \) is the set of the two neighbors of pixel \( i \) (boundary pixels have only one neighbor).
Figure 2:
Fig. 2: Cameraman image (300 × 300) (a) Original image. (b) Noisy image corrupted by speckle noise. (c) AOS algorithm: $PSNR = 26.36$, $\tau = 0.25$, $\lambda = 0.1$, $iterations = 8$. (d) PMS algorithm: $PSNR = 26.62$, $\tau = 0.15$, $\lambda = 0.1$, $iterations = 16$. 
Figure 3:

Fig. 3: Lena image (300 × 300) (a) Original image. (b) Noisy image corrupted by speckle noise. (c) AOS algorithm: $PSNR = 29.97$, $\tau = 0.15$, $\lambda = 0.7$, $iterations = 23$. (d) PMS algorithm: $PNSR = 29.65$, $\tau = 0.15$, $\lambda = 0.15$, $iterations = 34$. 
References


