Strongly Pure Fuzzy Ideals And Strongly Pure Fuzzy Submodules

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Abstract

The main aim of this paper is to extend and study the notion of (ordinary) S-pure ideal (submodule) into S-pure fuzzy ideal (submodule) and S-regular ring (module) into S-regular fuzzy ring (module). This lead us to introduced and study other notions such as S-pure fuzzy ideal (submodule) and S-regular fuzzy ring (module).

Introduction

Let I fuzzy ideal of a ring R. it is well known that I is called S-pure fuzzy ideal of R if for each $r_t \subseteq I$, there exists a prime fuzzy singleton $x_t \subseteq I$ such that $r_t = r_t x_t, \forall t, \ell \in (0,1]$.

And a fuzzy ring R is called S-regular fuzzy if and only if for each fuzzy singleton $r_t$ of $R$, there exists a prime fuzzy singleton $x_t$ of $R$ such that $r_t = r_t x_t r_t, \forall t, \ell \in (0,1]$.

In this paper, we fuzzify these concepts S-pure fuzzy ideal (submodule) and S-regular fuzzy ring (module), moreover we generalize many properties of S-pure fuzzy ideal (submodule) and S-regular fuzzy ring (module).

This paper consists of four parts. In part one, various basis properties about strongly pure fuzzy ideals are discussed. Part two included strongly regular fuzzy ring and basic properties about this concept. Part three study the strongly pure fuzzy submodules. Part four is definition the strongly regular fuzzy module and study the property of strongly regular fuzzy module.

§1. Strongly Pure Fuzzy Ideals

Definition (1.1)

Let I be a fuzzy ideal of a ring R, I is called pure fuzzy ideal if for each $x_t \subseteq I$, there exists $r_t \subseteq I$ such that $x_t = x_t r_t, \forall t, \ell \in (0,1]$.

Definition (1.2)

Let $x_t: R \rightarrow [0,1]$ such that $x_t(p) = \begin{cases} t & \text{if } x = p \\ 0 & \text{otherwise} \end{cases}$ Where p is prime number of R.
Definition (1.3)

Let $K$ be a fuzzy ideal of a ring $R$. $K$ is called strongly pure fuzzy ideal denoted by $S$-pure fuzzy ideal if for each $r_t \subseteq K$, there exists a prime fuzzy singleton $x_t \subseteq K$ such that $r_t = r_tx_t$, $\forall t, \ell \in (0,1]$.

Proposition (1.4)

Let $I$ be a fuzzy ideal of $R$ then $I$ is $S$-pure if and only if $I_t$ is a $S$-pure ideal of $R$, $\forall t \in (0,1]$.

Proof:

$(\Rightarrow)$ Let $r_t = r_tx_t$, $\forall t \in (0,1]$. To show that $r = rx$

$r_t = (rx)_t$ where $\ell = \min\{\ell, t\}$ by [1]

$r = rx$ by [4]

Then $I_t$ is $S$-pure ideal of $R$, $\forall t \in (0,1]$.

$(\Leftarrow)$ let $r = rx$ To prove $r_t = r_tx_t$, $\forall t \in (0,1]$

$r = rx$ implies $r_t = (rx)_t$ by [4]

$r_t = r_tx_t$ where $\ell = \min\{\ell, t\}$ by [1]

Therefore $I$ is a $S$-pure fuzzy ideal of $R$.

Definition (1.5)

An fuzzy singleton $a_t$ of $R$ is called fuzzy idempotent if $(a_t)^2 = a_t$, $\forall t \in (0,1]$.

Remarks and Examples (1.6)

1-Let $K$ be a fuzzy ideal of a ring $R$, if $K$ is $S$-pure fuzzy ideal then $K$ is pure fuzzy ideal.

Proof: it is clear

The converse not true by

Example: A ring $Z_6$ and $N=\{3\} = \{0, 3\}$, $K=\{2\} = \{0, 2, 4\}$

Define $K: Z_6 \rightarrow [0,1]$ by $I(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall t \in (0,1]$
Define $J: Z_6 \rightarrow [0,1]$ by $J(x) = \begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases} \forall t \in (0,1)$

It is clear that $K$ and $J$ is fuzzy ideal of $Z_6$ and $J_t = K, K_t = N$

$K_t$ is S-pure ideal of $Z_6$ by [5]

Thus $K$ is S-pure fuzzy ideal of a ring $Z_6$ by (Proposition 3.1.4)

But $J_t$ is not S-pure ideal of a ring $Z_6$ by [5]

Hence $J$ is not S-pure fuzzy ideal of $Z_6$ by (Proposition 3.1.4)

But $J$ is pure fuzzy ideal.

2- Let $K$ be a fuzzy ideal of a ring $R$. If $K$ is S-pure fuzzy ideal of $R$, then $JK = J \cap K$ for each fuzzy ideal $J$ of $R$.

3- Let $K$ be a fuzzy ideal of a ring $R$. If $K$ generated by prime idempotent fuzzy singleton, then $K$ is S-pure fuzzy ideal

Proof:

Let $K = (p_s)$ be a fuzzy ideal generated by prime fuzzy singleton $p_t, \forall t \in (0,1)$

Such that $p_s = p_s^2$. If $x_t \subseteq K$ there exists fuzzy singleton $r_t$ of $R$ such that $x_t = r_t p_s$ implies $x_t = r_t p_s = r_t p_s^2 = r_t p_s p_s = x_t p_s$

Therefore $K$ is S-pure fuzzy ideal of a ring $R$.

4- Let $K$ be a fuzzy ideal of a ring $R$, if $K$ is S-pure fuzzy ideal then $K$ is idempotent.

Proof:

Let $K$ be a S-pure fuzzy ideal of $R$ and let $r_t \subseteq K, \forall t \in (0,1)$

Then there exists a prime fuzzy $x_t \subseteq K$.

Such that $r_t = r_t x_t \forall t \in (0,1)$ but $r_t x_t \subseteq K.K$

Hence $r_t \subseteq K^2$. Thus $K \subseteq K^2$ and it is clear that $K^2 \subseteq K$. implies $K = K^2$.

Therefore $K$ is idempotent fuzzy ideal of $R$.

Definition (1.7)

Let $R$ be a fuzzy ring, then there exists $1_t$ of $R$ such that $a_t.1_t = (a.1)_t = a_t$ for all fuzzy singleton $a_t$ of $R$. $a_t$ is called unit fuzzy singleton.
Definition (1.8)

Let \( x_t \) be a fuzzy singleton of \( R \) is called fuzzy irreducible if \( x_t = r_t y_s \), where \( r_t \neq 0, y_s \) is a fuzzy singleton of \( R \) , \( \forall t, s, t \in (0,1] \) it is non unit fuzzy singleton of \( R \) then \( r_t \) or \( y_s \) is unity of \( R \).

Now, we introduce the concepts of fuzzy factorial ring

Definition (1.9)

Let \( S \) be a non empty fuzzy subset of \( R \), and has no fuzzy singlet unit of integral domain of \( R \), then \( R \) is called fuzzy factorial if every non empty fuzzy singleton of \( R \) written uniquely form \( y_r x_{t1} \ldots x_{tk} \) where \( y_r \) is unit of \( R \) and \( x_{t1} \ldots x_{tk} \subseteq S, \forall r, t \in (0,1] \).

Lemma (1.10)

Let \( R \) be a factorial fuzzy ring. Then every irreducible fuzzy singleton \( y_r \) of \( R \) is fuzzy prime every \( , x_t \subseteq S \) is prime fuzzy singleton and every prime fuzzy singleton of set \( S \) is the product of unit of \( R, \forall t \in (0,1] \).

Proof:

Let \( y_r \) irreducible fuzzy singleton of \( R \)
Thus \( y_r \) is non unit and if \( a_s b_r \subseteq (y_r) \)
Then \( a_s b_r = x_t y_r \) with \( x_t \subseteq S \), we write \( a_s, b_r, x_t \) as product of irreducible
\[
 a_s = y_{r1} \ldots y_{rl} \quad b_r = q_{k1} \ldots q_{km} \quad x_t = r_{t1} \ldots r_{tn} \quad \forall r, k, t \in (0,1].
\]
Here, one of those first two product may be empty
\[
 y_{r1} \ldots y_{rl} q_{k1} \ldots q_{km} = r_{t1} \ldots r_{tn} y_r
\]
It is mean that either \( a_s \subseteq (y_r) \) or \( b_r \subseteq (y_r) \)
Thus \( (y_r) \) is prime fuzzy ideal of \( R \) and it is generated by prime

Proposition (1.11)

Let \( K \) be fuzzy ideal of \( R \). And let \( R \) be a factorial fuzzy ring, such that \( y_r \neq 0, y_r \) is non unit fuzzy singleton of \( R \) is fuzzy irreducible. Then \( K \) is \( S \)-pure fuzzy ideal \( \iff \) \( K \) is pure fuzzy ideal.

Proof:

Let \( K \) be a pure fuzzy ideal of \( R \), and let \( r_t \subseteq K \), there exists \( x_t \subseteq K \), such that \( r_t = r_t x_t \), since \( x_t \subseteq R \), is irreducible fuzzy singleton of \( R \)
Hence \( x_t \) is prime of \( K \) by (lemma 3.1.10)
Therefore \( K \) is \( S \)-pure fuzzy ideal of \( R \).
The conversely is clear.

**Proposition (1.12)**

Let $K$ and $H$ are two fuzzy ideal of a ring $R$, if $K$ is $S$-pure fuzzy ideal of $R$ then $K \cap H$ is $S$-pure fuzzy ideal of $R$.

**Proof:** obviously.

**Proposition (1.13)**

Let $K$ and $H$ are two fuzzy ideal of a ring $R$, such that $K \subseteq H$ if $K \cap H$ is $S$-pure fuzzy ideal of $R$, then $K$ is $S$-pure fuzzy ideal of $R$.

**Proof:** it is clear.

**Corollary (1.14)**

Let $K$ and $H$ are two $S$-pure fuzzy ideal of a ring $R$, then $K \cap H$ is $S$-pure fuzzy ideal of $R$.

**Corollary (1.15)**

Let $K$ and $H$ are two fuzzy ideal of a ring $R$, then $K$ is $S$-pure fuzzy ideal of $R$ if and only if $K \cap H$ is $S$-pure fuzzy ideal of $R$.

**Proposition (1.16)**

Let $K$ and $H$ are two fuzzy ideal of a ring $R$, if $K \oplus H$ is $S$-pure fuzzy ideal of $R$. then either $K$ or $H$ is $S$-pure fuzzy ideal of $R$.

**Proof:**

Let $x_t \subseteq K$ and $y_t \subseteq H$, implies $x_t + y_t \subseteq K \oplus H$

Since $K \oplus H$ is $S$-pure fuzzy ideal of $R$, there exists a prime fuzzy $r_t \subseteq K \oplus H$

Where $r_t$ is prime $r_t + 0 \subseteq K \oplus H$

Such that $x_t + y_t = (x_t + y_t) r_t = (x_t + y_t)(r_t + 0) = x_t r_t + y_t r_t \subseteq K \oplus H$

Since $y_t r_t \subseteq K \cap J$ and $K \cap H = \{0\}$

Hence $y_t r_t = 0$. Thus $x_t = x_t r_t \subseteq K$.

Therefore $K$ is $S$-pure fuzzy ideal of $R$.

And if $r_t = 0 + r_t \subseteq K \oplus J$, then we can get $H$ is $S$-pure fuzzy ideal of $R$. 
Corollary (1.17)

Let $K$ and $H$ are two fuzzy ideal of $R$ and let $R$ be a factorial fuzzy ring, such that $K \oplus H$ is $S$-pure fuzzy ideal of $R$, then $K$ and $H$ are $S$-pure fuzzy ideal of $R$.

"Definition (1.18)"

The fuzzy Jacobson radical of a ring $R$ denoted by $F-J(R)$ is the intersection of all fuzzy maximal ideal of $R$.[3]"

Proposition (1.19)

Let $K$ is $S$-pure fuzzy ideal of $R$, such that $K \subseteq F-J(R)$, then $K=\{0\}$

Proof:

Let $r_\ell \subseteq K$, since $K$ is $S$-pure fuzzy ideal of $R$ there exists a prime $x_\ell \subseteq K$, such that $r_\ell = r_\ell x_\ell$

Implies $r_\ell (1-x_\ell)=0$. And since $K \subseteq F-J(R)$, then $x_\ell \subseteq F-J(R)$.

Hence $r_\ell=0$, so $K=\{0\}$.

§2. Strongly Regular Fuzzy Ring

Definition (2.1)

A fuzzy ring $R$ is called regular if and only if for each fuzzy singleton $x_\ell$ of $R$, there exists fuzzy singleton $r_\ell$ of $R$ such that $x_\ell = x_\ell r_\ell x_\ell$, $\forall \ell \in (0,1]$.

Definition (2.2)

Let $R$ be a fuzzy ring, $R$ is called strongly regular denoted by $S$-regular fuzzy if and only if for each fuzzy singleton $r_\ell$ of $R$, there exists a prime fuzzy singleton $x_\ell$ of $R$ such that $r_\ell = r_\ell x_\ell r_\ell$, $\forall \ell \in (0,1]$.

Equivalent a ring $R$ is $S$-regular fuzzy if for each fuzzy singleton of $R$ is $S$-regular fuzzy.

Proposition (2.3)

Let $R$ be a $S$-regular fuzzy ring $\iff R_t$ be $S$-regular ring, $\forall t \in (0,1]$.

Proof:

$(\Rightarrow)$ Let $R$ be a $S$-regular fuzzy ring. To prove $R_t$ is $S$-regular ring

$r_\ell = r_\ell x_\ell r_\ell$, $\forall \ell, t \in (0,1]$.

implies $r_\ell = (rxr)_\ell$ where $\ell = \min\{\ell, t\}$ by[1]

$r=rxr$ by[4]

Then $R_t$ is $S$-regular ring, $\forall t \in (0,1]$. 
(\Leftrightarrow) let R be S-regular ring to show that R is S-regular fuzzy ring

Let r=r \times r. To prove \( r_{\ell} = r_{\ell} \times r_{\ell} \) \( \forall \ell, t \in (0,1] \)

\( r_{\ell} = (r \times r)_{\ell} \) where \( \ell = \min \{ \ell, t \} \) by [4]

\( r_{\ell} = r_{\ell} \times r_{\ell} \) by [1]

Therefore R is S-regular fuzzy ring.

**Remarks and Examples (2.4)**

1- Let R: \( \mathbb{Z}_4 \rightarrow [0,1] \) define by

\[
R(r) = \begin{cases} 
 t & \text{if } r \in \mathbb{Z}_4 \\
 0 & \text{otherwise}
\end{cases} \ \forall \ t \in (0,1]
\]

It is clear that \( R_4 = \mathbb{Z}_4 \) and \( \mathbb{Z}_4 \) is S-regular ring [5]

Thus R is S-regular fuzzy ring. by proposition(3.2.3)

By the same method we can show that if \( R_6 = \mathbb{Z}_6 \) is not S-regular fuzzy ring and we get R is not S-regular fuzzy ring.

2- Let \( R \) be a fuzzy ring, if \( R \) is S-regular fuzzy, then \( R \) is regular fuzzy ring.

Proof: it is clear

The converse not true for example

**Example:** Let R: \( \mathbb{Z}_6 \rightarrow [0,1] \) define by

\[
R(r) = \begin{cases} 
 t & \text{if } r \in \mathbb{Z}_6 \\
 0 & \text{otherwise}
\end{cases} \ \forall \ t \in (0,1]
\]

It is clear that \( R_6 = \mathbb{Z}_6 \) and \( \mathbb{Z}_6 \) is regular ring, but \( \mathbb{Z}_6 \) is not S-regular ring

Then R is not S-regular fuzzy ring. by proposition(3.2.3)

3- Every fuzzy ideal of R is irreducible and let R be a factorial fuzzy ring, then R is S-regular fuzzy ring \( \Leftrightarrow \) fuzzy ideal of R is S-pure fuzzy.

**Proposition (2.5)**

Let \( R_1 \) and \( R_2 \) are two fuzzy ring, if \( R_1 \oplus R_2 \) is S-regular fuzzy ring. Then either \( R_1 \) or \( R_2 \) is S-regular fuzzy ring.

Proof:

Let fuzzy singleton \( r_{1\ell} \) of \( R_1 \) and \( r_{2\ell} \) of \( R_2 \)

Implies \( r_{1\ell} + r_{2\ell} \subseteq R_1 \oplus R_2 \). Put \( x_{\ell} = r_{1\ell} + r_{2\ell} \)
Since \( R_1 \oplus R_2 \) is S-regular fuzzy ring, there exists a prime fuzzy singleton \( y_t = y_t + 0 \leq R_1 \oplus R_2 \), such that
\[
x_t = x_t y_t x_t = (r_{t1} + r_{t2}) y_t (r_{t1} + r_{t2})
\]
\[
= r_{t1} y_t r_{t1} + r_{t2} y_t r_{t2} + r_{t1} y_t r_{t2} + r_{t2} y_t r_{t1}
\]
But \( r_{t1} y_t R_{t1}, r_{t2} y_t R_{t1} \subseteq R_1 \cap R_2 \) and \( R_1 \cap R_2 = (0) \)
Thus \( x_t = r_{t1} + r_{t2} = r_{t1} y_t r_{t1} + r_{t2} y_t r_{t2} \).

Imply that \( r_{t1} = r_{t1} y_t r_{t1} \subseteq R_1 \)
Therefore \( R_1 \) is S-regular fuzzy ring.

And if \( 0 + y_t \leq R_1 \oplus R_2 \), by same method we get \( r_{t2} = r_{t2} y_t r_{t2} \subseteq R_2 \)
Hence \( R_2 \) is S-regular fuzzy ring.

§3. Strongly Pure Fuzzy Submodules

**Definition (3.1)**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \) and let \( A \) be a fuzzy submodule of \( X \). \( A \) is called Strongly pure fuzzy submodule denoted by S-pure fuzzy submodule, if there exists a prime fuzzy ideal \( P \) of a ring \( R \) such that \( P X \cap A = PA \).

**Proposition (3.2)**

Let \( B \) be fuzzy submodules of a fuzzy module \( X \). Then \( B \) is S-pure fuzzy submodule of \( X \) if and only if \( B_t \) is S-pure submodules of \( X_t, \forall \ t \in (0,1] \).

**Proof:**

Let \( I \) be a prime ideal of ring \( R \).

Define \( P: R \to [0,1] \) by \( P(x) = \begin{cases} t & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases} \forall \ t \in (0,1] \)

And let \( N \) be a submodule of an \( R \)-module \( M \).

Define \( B: M \to [0,1] \) by \( B(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \forall \ t \in (0,1] \)

It is clear that \( P \) is a prime fuzzy ideal of \( R \) and \( B \) is fuzzy submodules of \( X \).

Now, \( B_t = N, \ P_t = I \), \( X_t = M \)

(\( \Rightarrow \)) Let \( B \) is S-pure fuzzy submodule of \( X \). To prove \( B_t \) is S-pure submodules of \( X_t, \forall \ t \in (0,1] \).

To show that \( P_t X_t \cap B_t = B_t P_t \)

\[
R X_t \cap B_t = (P X)_t \cap B_t \quad \text{by[2]}
\]
\[
= (P X \cap B)_t \quad \text{by[6]}
\]
Thus $B_t$ is $S$-pure submodules of $X_t, \forall t \in (0,1]$. Conversely, let $P$ be a prime fuzzy ideal of $R$ and $B$ be a fuzzy submodules of $X$.

$T_p B$ is $S$-pure fuzzy submodule of $X$

$$(PX \cap B)_t = (PX)_t \cap B_t \quad \forall t \in (0,1]. \quad \text{By[6]}$$

$$= P_t X_t \cap B_t \quad \text{by[2]}$$

but $B_t$ is $S$-pure submodules of $X_t$. Then $P_t X_t \cap B_t = P_t B_t$

$$= (PB)_t \quad \text{by[2]}$$

Hence $(PX \cap B)_t = (PB)_t$

$PX \cap A = PB$

Therefore $B$ is $S$-pure fuzzy submodule of $X$.

**Remarks and Examples (3.3)**

1- Let $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module and $N = (3)$ and $K = (2)$

Define $X: \mathbb{Z}_6 \rightarrow [0,1]$ by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define $A: \mathbb{Z}_6 \rightarrow [0,1]$ by $A(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$

Define $B: \mathbb{Z}_6 \rightarrow [0,1]$ by $B(x) = \begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$

It is clear that $X$ is fuzzy module, and $A, B$ is fuzzy submodules of $X$

$X_t = M, A_t = N$ and $B_t = K$

$A_t$ is $S$-pure submodules of $X_t$. by[5]

Then $A$ is $S$-pure fuzzy submodule of $X$ by(proposition 3.3.2)

But $B_t$ is not $S$-pure fuzzy submodule of $Z_6$

Therefore $B$ is not $S$-pure fuzzy submodule of $X$. by(proposition 3.3.2)

2- Let $X$ be a fuzzy module of an $R$-module $M$, and let $C$ be $S$-pure fuzzy submodule of $X$, then $C$ is pure fuzzy submodule of $X$. **
Proof: it is clear

The converse not true for example

Example: Let M=\( Z_{12} \) as Z-module and N= (3)

Define \( X: M \rightarrow [0,1] \) by \( X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases} \)

Define \( C: M \rightarrow [0,1] \) by \( A(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \) \( \forall t \in (0,1] \)

It is clear that \( X \) is fuzzy module, \( C \) is fuzzy submodules of \( X \), \( X_t=M \) and \( C_t=N \)

\( C_t \) is pure submodules of \( X_t \) by[18]

Thus \( C \) is pure fuzzy submodule of \( X \) by[4]

But \( C_t \) is not S-pure submodules of \( X_t \) by [5]

Therefore \( C \) is not S-pure fuzzy submodule of \( X \). by(proposition 3.3.2)

**Proposition (3.4)**

let \( A \) be fuzzy submodules of a fuzzy module \( X \) and let \( B \) be fuzzy submodules of \( A \). if \( A \) is pure fuzzy submodule of \( X \) and \( B \) is S-pure fuzzy submodules of \( A \), then \( B \) is S-pure fuzzy submodules of \( X \).

Proof:

Let \( B \) is S-pure fuzzy submodules of \( A \), there exists a prime fuzzy ideal \( P \) of \( R \) such that 
\( PA \cap B = PB \)

Since \( A \) is pure fuzzy submodule of \( X \), then \( PX \cap A = PA \).

Implies that \( B \cap A \cap XP = BP \), and since \( B \subseteq A \), then \( B \cap A = B \)

Implies \( XP \cap B = PB \)

Therefore \( B \) is S-pure fuzzy submodules of \( X \).

**Corollary (3.5)**

let \( A \) and \( B \) are two fuzzy submodule of a fuzzy module \( X \). if \( A \) is pure fuzzy submodule of \( X \) and \( A \cap B \) is S-pure fuzzy submodules of \( A \), then \( A \cap B \) is S-pure fuzzy submodules of \( X \).
§4. Strongly Regular Fuzzy Module

Definition (4.1)

Let $X$ be a fuzzy module of an $R$-module $M$, an fuzzy $x_t \subseteq X, \forall t \in (0,1]$ is called strongly regular fuzzy denoted by $S$-regular fuzzy if there exists a fuzzy module homomorphism $\theta: M \rightarrow R$, such that $\theta(x_t) x_t = x_t$ where $\theta(x_t)$ is $S$-regular fuzzy singleton in a ring $R$.

Definition (4.2)

Let $X$ be a fuzzy module of an $R$-module $M$, is called $S$-regular fuzzy if every $x_t \subseteq X, \forall t \in (0,1]$ is $S$-regular fuzzy

Remark (4.3)

Let $X$ is S-regular fuzzy modules and $A$ be a prime fuzzy submodule of $X$, then $A$ is $S$-regular fuzzy submodule of $X$.

Proof:

Let $A$ be a fuzzy submodule of $X$ and $I$ be a fuzzy ideal of $R$.

To show that $I X \cap A = IA$

It is clear that $IA \subseteq IX \cap A$

To prove $IX \cap A \subseteq IA$

Let $x_t \subseteq IX \cap A$, then $x_t = \sum_{i=1}^{n} r_{ti} x_{ti} \forall t \in (0,1]$. Where $r_{ti} \subseteq I$ and $x_{ti} \subseteq X$

Since $X$ is S-regular fuzzy $R$-modules , hence $x_t$ S-regular fuzzy singleton

Thus there exists a fuzzy module homomorphism $\theta: M \rightarrow R$, such that $x_t = \theta(x_t) x_t$, so $\theta(x_t) = \sum_{i=1}^{n} r_{ti} \theta(x_{ti})$ and $x_t = \theta(x_t) x_t = \sum_{i=1}^{n} r_{ti} \theta(x_{ti}) x_t$

And since $x_t \subseteq A$, hence $x_t = \sum_{i=1}^{n} r_{ti} \theta(x_{ti}) x_t \subseteq IA$

Thus $IX \cap A \subseteq IA$

Then $IX \cap A = IA$

Therefore $A$ is S-regular fuzzy submodule of $X$.

Proposition (4.4)

Let $R$ be a S-regular fuzzy ring $\iff R$ is S-regular fuzzy $R$-module.

Proof:

Let $R$ be a S-regular fuzzy ring to prove $R$ is S-regular fuzzy $R$-module.
Let $x_t \subseteq R$, thus there exists a prime fuzzy singleton $p_s \subseteq R$ such that $x_t = x_t p_s x_t, \forall t, \ell \in (0,1]$.

Now, define a function $\theta: R \rightarrow R$, by $\theta(x_t) = x_t p_s x_t$ for each fuzzy singleton $x_t$ of $R$.

Then $\theta(x_t) x_t = x_t p_s x_t, \forall t, \ell \in (0,1]$.

Thus $R$ is $S$-regular fuzzy $R$-module.

Conversely let $R$ is $S$-regular fuzzy $R$-module to prove $R$ is $S$-regular fuzzy ring.

Let $x_t \subseteq R$, there exists a fuzzy module homomorphism $\theta: R \rightarrow R$, such that $x_t = \theta(x_t) x_t$, where $\theta(x_t)$ is $S$-regular fuzzy singleton of $R$.

Since $\theta(x_t) = \theta(1) x_t$, hence $x_t p_s x_t = x_t \theta(1) x_t$.

Therefore $R$ is $S$-regular fuzzy ring.

**Proposition (4.5)**

Let $X$ is $S$-regular fuzzy module and fuzzy divisible over an fuzzy integral domain $R$, then every fuzzy submodule of $X$ is fuzzy divisible.

Proof:

Let $A$ be a fuzzy submodule of $X$, and $0 \neq r_t \subseteq R$.

We show that $r_t A = A$.

By remark(3.4.3) $A$ is $S$-pure fuzzy submodules of $X$.

so $<r_t> A = A \cap <r_t> X$.

We show that $r_t A = A \cap r_t X$. if $x_t \subseteq A \cap r_t X$, then $x_t = r_t s_t$.

Since $x_t$ is $S$-regular fuzzy of $X$ there exists a fuzzy module homomorphism $\theta: M \rightarrow R$, such that $x_t = \theta(x_t) x_t$ and hence $x_t = \theta(x_t) x_t = r_t \theta(s_t) x_t$, as $x_t \subseteq A$.

This implies that $x_t \subseteq r_t A$, hence $A \cap r_t X \subseteq r_t A$.

Hence $r_t A = A \cap r_t X$.

As $r_t X = X$, thus $A \cap X = r_t A$.

So $r_t A = A$.

Therefore $A$ is fuzzy divisible.
Proposition (4.6)

Let \( X \) be a \( S \)-regular fuzzy \( R \)-module \( M \), then \( F-J(R) X = 0 \)

\( \text{Proof:} \)

Since \( X \) is \( S \)-regular fuzzy \( R \)-module,

Thus every fuzzy submodules of \( X \) is \( S \)-pure fuzzy submodules by remark(3.4.3)

Let \( F-J(R) X \neq 0 \), therefore there exists \( x_t \subseteq F-J(R) \)

So \( R x_t \) is \( S \)-pure fuzzy submodules of \( X \)

Thus \( R x_t \cap F-J(R) X = F-J(R) R x_t \)

Then \( R x_t = F-J(R) R x_t \) and by lemma(3.4.6) \( R x_t = 0 \) so \( x_t = 0 \)

Hence \( F-J(R) X = 0 \)

REFERENCES


