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# $S_{gs}^*$-$T_i$ Spaces and $S_{gs}^*$-Urysohn Spaces , Where i=1,2

by

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**Abstract**

In this paper , we will define and study a new types of separation axioms we called "$S_{gs}^*$-$T_0$-space, $S_{gs}^*$-$T_1$-space, $S_{gs}^*$- $T_2$-space,", $S_{gs}^*$- regular spaces and $S_{gs}^*$- normal spaces in topological spaces, also some important propositions will be given and some properties will be studied, as well as we study some characters of these concepts and the relationship among them.

## 1.Introduction:

Khalaf [3] introduced and studied the properties of semi-separation axioms. The notion of $S_{gs}^*$-open set [5]introduced by B.J.Tawfeeq in 2016. Some new separation axioms called semi-$T_i$, (i=0, 1, 2) spaces introduced by S.N.Maheshwari and R. Prasad [4], A. Kar and P. Bhattacharyya [2] defined and studies new types of separation axioms called pre-$T_i$-spaces for i=0, 1, 2 with some characterization of this concept.

In this paper we define and characterized new types of separation axioms called $S_{gs}^*$ - $T_i$ Spaces where i=0, 1, 2 , $S_{gs}^*$- Urysohn Spaces, $S_{gs}^*$- regular spaces and $S_{gs}^*$- normal spaces in topological spaces.

Also we define new types of separation axioms called stronger semi $S_{gs}^*$-$T_i$ spaces where (i=0,1,2) with basic properties of this concept.

## 2.Basic concepts and Preliminaries

Before entering into our work we recall the following definitions which are useful in the following sections.

**Definition 2.1**:**[5]:**A semi open set A of topological space $(X,\tau)$ is said to be $S_{gs}^*$-open , if for each x∈A, there exist a $gs^*$ -closed set F such that x∈ F⊆A.

A subset A of a space X is $S_{gs}^*$-closed set **,**if X- A is an $S_{gs}^*$-open set.

**Proposition 2.2: [5]** A subset A of a space X is $S_{gs}^*$-closed set if and only if A is semi- closed set and it is an intersection of $gs^*$ open sets

 

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The family of all $\mathbf{S_{gs}}^*$-open subsets of X is denoted by $\mathbf{S_{gs}}^*$-(W,S ) or $\mathbf{S_{gs}}^*$-O( X).

**Definition 2.3:[5]** Let A be a subset of a topological space (X,τ). A point x ∈X is said to be $\mathbf{S_{gs}}^*$-limit point of A if for each $\mathbf{S_{gs}}^*$-open set U containing x ,we have (U-{x} ) ∩ A $\neq \varphi$.

The set of all $\mathbf{S_{gs}}^*$-limit point of A is said to be $\mathbf{S_{gs}}^*$-derive set of A and is denoted by $D_{Sgs*}$ (A).

**Definition 2.4:[2],[4],** A space(X,τ) is said to be:
1) Semi-$T_0$ (resp., strongly semi-$T_0$) space if for each two distinct points and in X, there exists a semi-open (resp.,$\theta$-semi-open) set containing one of them but does not contain the other.

2) Semi-$T_1$ (resp., strongly semi-$T_1$ ) space if for each two distinct points a and b in X, there exist semi-open (resp.,$\theta$ -semi-open) U and V sets containing a and b respectively, such that a ∉U and b ∉V

3) semi-$T_2$ (resp., strongly semi-$T_2$ ) space if for each two distinct points a and b in X, there exist two disjoint semi-open (resp., $\theta$-semi-open) U and V sets and containing x, y respectively

**Lemma 2.5:[1]** A space X is semi-$T_1$ , if and only if {x} is semi closed for any point x ∈ X.

**Proposition 2.6:[5]** If a space X is semi-$T_1$, then $\mathbf{S_{gs}}^*$ O(X) = SO(X).

**Proposition 2.7: [5]**

1. Every semi-$\theta$ -open subset of a space X is $\mathbf{S_{gs}}^*$-open.

2. Every $\theta$ -open subset of a space X is $\mathbf{S_{gs}}^*$-open.

3. Every $\theta$ - semi -open subset of a space X is $\mathbf{S_{gs}}^*$-open.

**Proposition 2.8:[5]** The set U is $\mathbf{S_{gs}}^*$-open set in the space X, if and only if for each x∈ G there exists an $\mathbf{S_{gs}}^*$-open set B such that x∈ B ⊆ U.

Now ,we introduce the following definitions:

## 3. Basic properties of $S_{gs}^*$-Separation Axioms

**Definitions 3.1:**

Let $(W,S)$ be topological space, then $W$ is said to be:

1. $\mathbf{S_{gs}}^*$-$T_0$ if given any two distinct points x,y of $W$,there is a $\mathbf{S_{gs}}^*$-open set which contain one of these point but not the other.

 

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**2.** $\mathbf{S_{gs}}^*$ -$T_1$ if given any two distinct points x,y of $W$,there are two $\mathbf{S_{gs}}^*$ -open sets G,H such that x∈ G but y∉ G and y∈ H but x ∉G.

**3.** $\mathbf{S_{gs}}^*$-$T_2$if given any two distinct points x,y of $W$, there are two disjoint $\mathbf{S_{gs}}^*$-open sets G,H such that x∈ Gand y∈$H$.

We say that $\mathbf{S_{gs}}^*$-$T_2$space is called $\mathbf{S_{gs}}^*$-Hausdorff space.

**4**. $\mathbf{S_{gs}}^*$-regular space, if given any closed F subset of $W$ and any point x of $W$ which is not in F ,there are two disjoint $\mathbf{S_{gs}}^*$-open sets G and H such that x∈$G$ and F⊂ H.

**5.** $\mathbf{S_{gs}}^*$- normal space, if given any two disjoint closed sets $F_1$,$F_2$of$W$,there are disjoint two $\mathbf{S_{gs}}^*$-open sets G,H such that $F_1$⊂G,$F_2$ ⊂ H.

**6.**strongly $\mathbf{S_{gs}}^*$-normal space, if given any two disjoint $\mathbf{S_{gs}}^*$-closed sets $F_1$,$F_2$ of $W$, there are two disjoint $\mathbf{S_{gs}}^*$-open sets G,H such that $F_1$⊂G, $F_2$ ⊂ H.

**7.** $\mathbf{S_{gs}}^*$**-**Urysohn Space if given any two distinct points x,y of $W$, there are two $\mathbf{S_{gs}}^*$-open sets G,H such that x∈ G and y∈$H$ and $\mathbf{S_{gs}}^*$cl(H) ∩ $\mathbf{S_{gs}}^*$ cl(G) =$\varphi$

**8.** sg$^*$ -$T_1$ if given any two distinct points x, y of $W$, there are two sg$^*$-open sets G,H such that x∈ G but y∉ G and y∈ H but x ∉ G.

**<u>Proposition 3.2:</u>** Every $\mathbf{S_{gs}}^*$-$T_1$ space is $\mathbf{S_{gs}}^*$-$T_0$ space.

**Proof:** Let ($W$,$S$) be topological space and let $x_1$,$x_2$ are two distant points of $W$

Since ($W$,$S$) is $\mathbf{S_{gs}}^*$-$T_1$ space, then there exists two $\mathbf{S_{gs}}^*$-open sets G,H such that $x_1$∈$G$, but $x_1$∉ H and $x_2$∈H but $x_2$∉ G .Then there is enough condition to get $\mathbf{S_{gs}}^*$-$T_0$ space.

**<u>Proposition 3.3:</u>** Every $\mathbf{S_{gs}}^*$-$T_2$ space is $\mathbf{S_{gs}}^*$-$T_1$ space.

**Proof:** Let ($W$,$S$) be $\mathbf{S_{gs}}^*$-$T_2$ space and let x,y are two distant points of $W$.

Since the space is $\mathbf{S_{gs}}^*$-$T_2$ space ,then there exists two $\mathbf{S_{gs}}^*$-open sets N, M such that x ∈ N, y∈M and M∩N=$\varphi$.

This implies that x∈ N but y ∉N and y∈M but x∉M. Hence the space is $\mathbf{S_{gs}}^*$-$T_1$ space.

**<u>Remark 3.4:</u>** The converse of Proposition 3.3, is not true in general as it is shown in the following example.

1.Let $W$={a,b,c,d} and$S$ ={$W$,φ, {a} ,{b}, {a,b},{a,b,c},{a,b,d}}
$\mathbf{S_{gs}}^*$O($W$)= {$W$,φ, {a} ,{ b,c} , { c,d} ,{a,c,d}},we get is $\mathbf{S_{gs}}^*$-$T_0$, but not $\mathbf{S_{gs}}^*$-$T_1$.

2.Let $W$be any infinite set with the co-finite topology. Then $W$is $T_1$-space, so by

 

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Proposition 2.6 $W$ is both semi-$T_1$ and $S_{gs}^*$-$T_1$ but is not $S_{gs}^*$-$T_2$ space.

**Remark 3.5:** Every $S_{gs}^*$-$T_i$ space is semi-$T_i$, for i=0, 1, 2. But the converse is not true in general as it is shown by the following examples:

**Example 3.6:**

**1.** Let $W = \{a, b, c\}$ and $S = \{W, \varphi, \{b\}, \{b,c\}\}$, then $SO(W) = \{W, \varphi, \{b\}, \{a,b\}, \{c,d\}\}$ and $S_{gs}^*O(W) = \{W, \varphi, \}$, we have $W$ is semi-$T_0$ space but not $S_{gs}^*$-$T_0$.

2. Let $W=\{a,b,c,d\}$ and $S = \{W, \varphi, \{b\}, \{d\}, \{b,d\}, \{a,b,d\}\}$, $SO(W) = \{W, \varphi, \{b\}, \{d\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$

$S_{gs}^*O¿) = \{W, \varphi, \{a,b,c\}, \{a,b,d\}\}$ this implies that $W$ is semi-$T_0$ but not $S_{gs}^*$-$T_1$ also is semi-$T_2$ but not $S_{gs}^*$-$T_2$.

**Proposition 3.7**: Every strongly semi-$T_0$ (resp., strongly semi-$T_1$, strongly semi-$T_2$) space is $S_{gs}^*$-$T_0$ (resp., $S_{gs}^*$-$T_1$, $S_{gs}^*$-$T_2$) space

**Proof:** Suppose that $(W,S)$ is $S_{gs}^*$- $T_0$ space and n, m are two distant points in $W$, then there exists an $\theta$ -semi-open set $M$ containing one of them but not the other, by Proposition 2.7, we get $M$ is $S_{gs}^*$- open set containing one of them but not the other. Therefore $W$ is $S_{gs}^*$-$T_0$

The converse of above proposition is not true in general as it is shown by the following example:

**Example 3.8:** Let $W=\{a,b,c,d\}$ and $S=\{W, \varphi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,d\}$, $SO(W) = \{W, \varphi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,c\} \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$, $S_{gs}^*O(W) = \{W, \varphi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,c\} \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}\}$, $\theta SO(W) = \{W, \varphi, \{a\}, \{b\}\}$, this implies that $W$ is $S_{gs}^*$-$T_o$ space but not strongly semi-$T_0$ space.

**Proposition 3.9:** $(W,S)$ is $S_{gs}^*$ -$T_1$ space if and only if $\{p\}$ $S_{gs}^*$-closed for each $p \in X$.

**Proof:** Suppose that $(W,S)$ is $S_{gs}^*$- $T_1$ space and let p be any point of X. We want to show that $\{p\}^c$ is $S_{gs}^*$-open for each $p \in X$.

Let $x \in \{p\}^c$, since $(X,\tau)$ is bg- $T_1$ space and $x \neq y$, so there is a $S_{gs}^*$-open set $G_x$ such that $x \in G_x$ but $p \notin G_x$.

Hence $x \in G_x \subset \{p\}^c$, and $\{p\}^c = \cup\{G_x : x \in \{p^c\}$. But the union of $S_{gs}^*$-open sets is $S_{gs}^*$-open set. Hence $\{p\}$ is $S_{gs}^*$-closed set.

Conversely, suppose that $\{p\}$ $S_{gs}^*$-closed for every $p \in X$ and let $a,b \in X$ such that $a \neq b$ then by hypothesis $\{a\}$, $\{b\}$ are disjoint $S_{gs}^*$-closed sets .

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This means that X-{a},X-{b} are $\mathbf{S_{gs}}^*$-open sets. Write G= X-{a} , H= X-{b}.We get b∈ G but a∉ G also a∈ H but b∉H. Therefore $(X,\tau)$ is $\mathbf{S_{gs}}^*$-$T_1$ space.

**<u>Proposition 3.10</u>:** If $(W,S)$ is $\mathbf{S_{gs}}^*$-$T_1$space , then it is sg$^*$-$T_1$.

**Proof:** Let $(W,S)$ is $\mathbf{S_{gs}}^*$-$T_1$ space and e ,f be two distant points in$W$, then there there are $\mathbf{S_{gs}}^*$-open set $M,Z$ such that e$\in M$ but f$\notin M$ and f$\in Z$ but e $\notin Z$.This gives by Definition 2.1, there exist two sg$^*$-closed sets $F_1$,$F_1$such thate$\in F_1 \subseteq M$ andf$\in F_2 \subseteq Z$ also we get $W$ -$F_1$,$W$ -$F_2$are gs$^*$- open sets such that e$\in W$ -$F_1$ but f$\notin W$ -$F_1$ and f $\in W$ -$F_2$ but e $\notin W$ -$F_2$. Thus $(W,S)$ is sg$^*$-$T_1$ space

The converse of above Proposition is not true in general as it is shown by the following example:

**<u>Example 3.11</u>:** Let $W$ = {a, b, c} and $S$ = {$W$,φ, {a,b},{a, b,c} },then SG$^*$O($W$)= {$W$,φ, {a} ,{b},{a,b},{a,b,c}{a,b,d}}and $\mathbf{S_{gs}}^*$ -O($W$)={$W$,φ, },we have $W$ is sg$^*$-$T_1$ space but not $\mathbf{S_{gs}}^*$-$T_1$.

**<u>Proposition 3.12</u>:** Every singleton set in $\mathbf{S_{gs}}^*$-Hausdorff space is $\mathbf{S_{gs}}^*$-closed sets .

**Proof:** Let $(X,\tau)$ ) is $\mathbf{S_{gs}}^*$-Hausdorff space, and x∈X. Since X is $\mathbf{S_{gs}}^*$-$T_2$ space, so by Proposition 3.10, X is $\mathbf{S_{gs}}^*$-$T_1$ space, and by Proposition 3.9 ,we obtain {x} is $\mathbf{S_{gs}}^*$-closed sets.

**<u>Proposition 3.13:</u>** A space $(W,S)$ is $\mathbf{S_{gs}}^*$-$T_1$ space if and only if for each x $\in W$ , $\mathbf{S_{gs}}^*$cl{x}={x}.

**Proof:** Suppose $(W,S)$ ) is $\mathbf{S_{gs}}^*$-$T_1$ space and x $\in W$ such that y ∈ $\mathbf{S_{gs}}^*$cl{x}

Then every $\mathbf{S_{gs}}^*$-open of y must contain x. Since $(W,\tau)$ ) is $\mathbf{S_{gs}}^*$-$T_1$ space, then there is no $\mathbf{S_{gs}}^*$-open of y which excludes x, this is contradiction with hypothesis.

Therefore $\mathbf{S_{gs}}^*$cl{x}={x} for all x $\in W$ .

Conversely, Suppose that $\mathbf{S_{gs}}^*$cl{x} ={x} for each x $\in W$ and that y,z are distinct points of $W$. If every $\mathbf{S_{gs}}^*$-open of y contains z ,then z ∈ $\mathbf{S_{gs}}^*$-cl{y}={y},hence y=z this is contradiction.

Therefore there is $\mathbf{S_{gs}}^*$-open of y which excludes z. Similarly there is $\mathbf{S_{gs}}^*$-open of z which does not y. Hence $(W,S)$ ) is $\mathbf{S_{gs}}^*$-$T_1$ space .

**<u>Proposition 3.14:</u>** A topological space $(W,S)$ is $\mathbf{S_{gs}}^*$-$T_1$ space if and only if $D_{Sgs*}$\{x\}=$\varphi$ for each x $\in W$

**Proof:** Let $(W,S)$ be a $\mathbf{S_{gs}}^*$-$T_1$ space, we get {x} is $\mathbf{S_{gs}}^*$-closed sets for all x $\in W$ , and $D_{Sgs*}$\{x\}⊆ {x},that mean $D_{Sgs*}$\{x\}⊆{x} or $D_{Sgs*}$\{x\}= $\varphi$.

If $D_{Sgs*}$\{x\}⊆{x}then for each $\mathbf{S_{gs}}^*$-open set U containing {x}we have

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(U-{x}∩ {x}) $\neq \varphi$ and this is contradiction. Therefore $D_{Sgs^*}$ {x}=$\varphi$.

Conversely, Suppose that $D_{Sgs^*}$ {x}= $\varphi$ .Now to prove $\mathbf{S_{gs}}^*$-cl{x} ={x}.

$\mathbf{S_{gs}}^*$-cl{x} ={x} ∪ $D_{Sgs^*}${x}={x} ∪$\varphi$ =$\varphi$ is bg-closed sets. Hence $(W,\tau)$ is $\mathbf{S_{gs}}^*$-$T_1$space .

**Proposition 3.15:**Let $(W,S)$ be a $\mathbf{S_{gs}}^*$- $T_0$ space then $\mathbf{S_{gs}}^*$cl{x}≠ $\mathbf{S_{gs}}^*$cl{y} for all x≠y

**Proof:** let $x_1$,$x_2$ are two distant points of. Since the space is $\mathbf{S_{gs}}^*$- $T_0$ space ,then there is an $\mathbf{S_{gs}}^*$-open set N which contain one of these point but not the other. Suppose x ∈N. It follows that x ∉ $\mathbf{S_{gs}}^*$cl{y}

**Definition 3.16:** A map f : $(W,S)\rightarrow$ (Y,$\tau$) is said to be (1-1) point $\mathbf{S_{gs}}^*$-closure if and only if for every x,y∈$W$ such that $\mathbf{S_{gs}}^*$cl{x}≠ $\mathbf{S_{gs}}^*$cl{y},then $\mathbf{S_{gs}}^*$cl (f{x})≠ $\mathbf{S_{gs}}^*$cl(f{y}) .

**Proposition 3.17:** If $(W,S)$ be a $\mathbf{S_{gs}}^*$- $T_0$ space and f : (X,$\tau$)→ (Y,$\tau$) is (1-1) point $\mathbf{S_{gs}}^*$-closure, then f is injective.

**Proof:** Suppose $(W,S)$ ) is $\mathbf{S_{gs}}^*$- $T_0$ space and x ,y are distinct points of $W$, by Proposition 3.14, we get $\mathbf{S_{gs}}^*$cl{x}≠ $\mathbf{S_{gs}}^*${y}. Since f is (1-1) point $\mathbf{S_{gs}}^*$- closure this implies that $\mathbf{S_{gs}}^*$cl (f{x})≠ $\mathbf{S_{gs}}^*$cl(f{y}) ,therefore f(x)≠ f(y).Thus f is injective.

**Definition 3.18:** A map f :$(W,S)\rightarrow$ (Y,$\tau'$) is said to be $\mathbf{S_{gs}}^*$-homeomorphism if :

1) f is bijective .

2) f is $\mathbf{S_{gs}}^{*-}$ irresolute .

3) $f^{-1}$is $\mathbf{S_{gs}}^*$-irresolute .

**Remark 3.19:** A property of a set which is preserved by $\mathbf{S_{gs}}^*$- homeomorphism is called $\mathbf{S_{gs}}^*$- topological property.

**Proposition 3.20:**The property of a space being $\mathbf{S_{gs}}^*$-$T_0$ space is a $\mathbf{S_{gs}}^*$-topological property.

**Proof:** Let f: $(W,S)\rightarrow$ (Y,$\tau'$) be $\mathbf{S_{gs}}^*$-homeomorphism and let (X,$\tau$) be $\mathbf{S_{gs}}^*$-$T_0$ space .

Given n,m are two distant points of Y. Since f is injective, there are $p,q\in W$ such that f(p) =n, f(q) =m ,since X is $\mathbf{S_{gs}}^*$-$T_0$ space then there is $\mathbf{S_{gs}}^*$-open sets $\omega$ such that p∈ $\omega$ , q∉ $\omega$ then n=f(p) ∈ f($\omega$) ,m=f( q)∉$f(\omega)$. f is $\mathbf{S_{gs}}^*$-homeomorphism and $\omega$ is an $\mathbf{S_{gs}}^*$-open subset of X, so f($\omega$) is a $\mathbf{S_{gs}}^*$-open subset of Y containing n but not m. Therefore (Y,$\tau'$) is $\mathbf{S_{gs}}^*$-$T_0$ space .

**Proposition 3.21 :**

**1.**The property of a space being $\mathbf{S_{gs}}^*$-$T_1$ space is a $\mathbf{S_{gs}}^*$-topological property.

2.The property of a space being $\mathbf{S_{gs}}^*$-$T_2$ space is a $\mathbf{S_{gs}}^*$-Topological property.

**Proof:** Let $(W,S)$ be $\mathbf{S_{gs}}^*$-$T_1$ space and let f: $(W,S) \to (Y,\tau')$ be $\mathbf{S_{gs}}^*$-homeomorphism

Since $(W,S)$ is $\mathbf{S_{gs}}^*$-$T_1$ space, given two distant points p,q $\in W$, then there are two $\mathbf{S_{gs}}^*$-open sets G ,H such that p∈G, q∉G and q∈H, p∉H.

Since f is injective then there are distant points $p_1$ ,$q_1$∈ X such that f(p) =$p_1$, f(q) =$q_2$, $p_1$, $q_2$∈ Y such that f(p) ≠f(q) .

Since $(W,S)$ is $\mathbf{S_{gs}}^*$-$T_1$ space ,then there exist two $\mathbf{S_{gs}}^*$-open sets f (G) ,f(H) such that

$p_1$=f(p)∈f (G) and $q_1$=f( q)∉f (G) and $q_1$=f( q))∈f (H) , $p_1$=f(p) ∉f (H)

Thus $(Y,\tau')$ is $\mathbf{S_{gs}}^*$-$T_1$ space .

**2.** Obvious.

**Proposition 3.22 :**

For any space$W$ the following statements are equivalent:
1. $W$is $\mathbf{S_{gs}}^*$-$T_1$ space.
2. Each subset of$W$ is the intersection of all $\mathbf{S_{gs}}^*$-open sets containing it.
3. The intersection of all $\mathbf{S_{gs}}^*$-open sets containing the point p$\in W$is the set {p}.

**Proof:1→2** Suppose that$W$is $\mathbf{S_{gs}}^*$-$T_1$ and $B \subseteq W$,that means for each z$\notin B$ , there exists a set $W-\{z\}$such that $B \subseteq W-\{z\}$and by Proposition 3.5, which implies that the set $W-\{z\}$is $\mathbf{S_{gs}}^*$-open for every z, it follows that$B=\cap\{W-\{z\}:z\in W\text{-}\{B\}\}$.We obtain that the intersection of all $\mathbf{S_{gs}}^*$-open sets containing $B$is $B$itself.
**2→3** Suppose that p$\in W$, then $\{p\}\in W$.By (2), we get the intersection of all $\mathbf{S_{gs}}^*$-open sets containing$\{p\}$is$\{p\}$itself. Then the intersection of all $\mathbf{S_{gs}}^*$-open sets containing p is $\{p\}$.
**3→1** Let $g,e\in W$such that $g\neq e$ ,we have by (3), the intersection of all $\mathbf{S_{gs}}^*$-open sets containing $g$and$e$ are $\{g\}$and$\{e\}$ respectively, then for each$g\in W$ there exists an $\mathbf{S_{gs}}^*$-open set M such that $g\in M$and $e\notin M$. Similarly for$e\in W$there exists an $\mathbf{S_{gs}}^*$ -open set N such that $e\in N$ and$g\notin N$.Thus$W$is $\mathbf{S_{gs}}^*$-$T_1$space.

**Proposition 3.23 :** If every finite subset of a space is $\mathbf{S_{gs}}^*$-closed , then is $\mathbf{S_{gs}}^*$-$T_1$ space.
**Proof:** Suppose that n ,m be two distant points in $W$. By hypothesis {n} and {m} are $\mathbf{S_{gs}}^*$-closed sets .This gives $W-\{n\}$ and $W-\{m\}$ are $\mathbf{S_{gs}}^*$-open sets such that n$\in W-\{m\}$ and m$\in W-\{n\}$ . This implies that $W$is $\mathbf{S_{gs}}^*$-$T_1$space.

**Proposition 3.24 :** lf $(W,S)$ is an $\mathbf{S_{gs}}^*$-$T_1$ space, then the topological space $W\times W$ is also $\mathbf{S_{gs}}^*$-$T_1$ space.

**Proof:** Let ($x_1$, $x_2$), ($y_1$, $y_2$)$\in W\times W$ be two distant points in $W\times W$, we get $x_1\neq y_1$ or $x_2\neq y_2$ , suppose that $x_2\neq y_2$. Since$W$is $\mathbf{S_{gs}}^*$-$T_1$ space then, there are two $\mathbf{S_{gs}}^*$ -open sets G,H such that $x_2$∈ G but $y_2$∉ G and $y_2$∈ H but $x_2$∉ G. We obtain that the sets G$\times W$

and $H \times W$ are $S_{gs}^*$ -open set in $W \times W$, we have $(x_1,x_2) \in G \times W$ and $(y_1,y_2) \in H \times W$. Since $W$ is $S_{gs}^*$-$T_1$ space ,we have $(x_1, x_2) \in G \times W$ and $(x_1, x_2) \notin H \times W$. Thus $W \times W$ is $S_{gs}^*$-$T_1$ space.

**Proposition 3.25 :** A space $(W,S)$ is an $S_{gs}^*$-$T_2$ space if and only if for each distinct points r, s of $W$ there exists an $S_{gs}^*$-neighborhood B of r such that $s \notin S_{gs}^*cl(B)$.
**Proof:** Let $W$ be an $S_{gs}^*$-$T_2$ space and let $r \in W$ , we get for each distinct points r, s of $W$ there exist two disjoint $S_{gs}^*$-open sets $A$ and $B$ such that $r \in A$ and $s \in B$. This mean that $r \in A \subseteq W - B$ and $W - B$ is an $S_{gs}^*$-neighborhood of r ,so it is $S_{gs}^*$-closed set in $W$ and $s \notin W - B$. Therefore $s \notin S_{gs}^*cl(W - B)$.
Conversely, given any two distinct points r,s of $W$, then by hypothesis, there exists an $S_{gs}^*$-neighborhood B of r such that $s \notin S_{gs}^*cl(B)$, so $s \in W - S_{gs}^*cl(B)$ and $r \notin W - S_{gs}^*cl(B)$. But $W - S_{gs}^*cl(B)$ is $S_{gs}^*$-open set. Since B is $S_{gs}^*$-neighborhood of r, then there exists an $S_{gs}^*$-open set $M$ of $W$ such that $r \in M \subseteq B$,we obtain $M \cap S_{gs}^*cl(B))=\varphi$. Thus $W$ is $S_{gs}^*$-$T_2$

**Proposition 3.26 :** If $(W,S)$ is an $S_{gs}^*$-Urysohn space, then the topological space X is also $S_{gs}^*$-$T_1$space.

**Proof:** Let $x, y \in X$ with $x \neq y$, since X is $S_{gs}^*$-Urysohn Space, by Definition 3.1 (7), there are two $S_{gs}^*$-open sets G,H such that $x \in G$ and $y \in H$ and $S_{gs}^*cl(H) \cap S_{gs}^*cl(G)= \varphi$, which implies that $x \notin S_{gs}^*cl(H)$ and $y \notin S_{gs}^*cl(G)$, it follows that $S_{gs}^*cl(H)$ , $S_{gs}^*cl(G)$ are $S_{gs}^*$-closed set in X .Now we have X- $S_{gs}^*cl$ (G) and X- $S_{gs}^*cl(H)$ are $S_{gs}^*$-open set such that $x \in S_{gs}^*cl(H)$ and $y \in S_{gs}^*cl(G)$, also $x \notin S_{gs}^*cl(G)$ and $y \notin S_{gs}^*cl(H)$.Therefore X is $S_{gs}^{¿}$-$T_1$space.

**Proposition 3.27:** A topological space X is an $S_{gs}^*$-$T_2$ space(resp. $S_{gs}^*$-$T_1$-Y space, $S_{gs}^*T_o$ -Y space), if $a,b \in X$ with $a \neq b$, there exists $S_{gs}^*$-continuous function $f:X \rightarrow T_2$ -Y space (resp.$T_1$-Y space, $T_0$-Y space),such that $f(a) \neq f(b)$.

**Proof:** Suppose that $a, b \in X$ with $a \neq b$ and $f:X \rightarrow T_2$ space Y such that $f(a) \neq f(b)$.Then there are disjoint open sets G,H of Y such that $f(a) \in H$ and $f(b) \in G$. Since f is $S_{gs}^*$-continuous function .This indicates that $f^{-1}(H)$,$f^{-1}(G)$ are disjoint open sets in X such that $a \in f^{-1}(H)$, $b \in f^{-1}(G)$. Hence X is an $S_{gs}^*$-$T_2$space.

**Proposition 3.28:** If $(W,S)$ is an $S_{gs}^*$-$T_2$ space, then the intersection of all $S_{gs}^*$-clopen sets of each point in $W$ is singleton.
**Proof:** Suppose that $(W,S)$ $S_{gs}^*$-$T_2$space and $\cap\{\beta:\beta$ is $S_{gs}^*$-clopen and $r \in \beta\}$ $=\{r,s\}$ where r,s are two distinct points ,then by hypothesis, there exists two disjoint $S_{gs}^*$-open sets $A$ and $B$ such that $r \in A$ and $s \in B$.It follows $r \in A \subseteq W - B$ .By Proposition 2.8, $W - B$ is $S_{gs}^*$-open set and furthermore, $W - B$ is $S_{gs}^*$-closed set ,this means that $W - B$ is $S_{gs}^*$-clopen containing r but not s which is a contradiction. Therefore the intersection of all $S_{gs}^*$-clopen sets containing r is {r }.

**Proposition 3.29:** A space $(W,S)$ is $S_{gs}^*$-$T_2$ if and only if given any two distinct points $u,v$ of $W$, there is a $S_{gs}^*$- clopen set which contain one of these point but not the other.

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**Proof:** Let $W$be $\mathbf{S_{gs}}^*$-T₂ space and $u, v$ any two distinct points of $W$ ,we obtain there exists two disjoint $\mathbf{S_{gs}}^*$-open sets $M$ and $N$such that $u \in M$ and $v \in N$. Since $M$ $\mathbf{S_{gs}}^*$-closed set, since is $\mathbf{S_{gs}}^*$-$T_2$ space, then for each t$u \in W - N$here exists an $\mathbf{S_{gs}}^*$-open set $Z$ such that $u \in Z \subseteq W - N$. By Proposition 2.8, $W - N$ is $\mathbf{S_{gs}}^*$-open set. This implies that $W - N$is $\mathbf{S_{gs}}^*$-clopen set.

**Conversely:** Suppose that for each distinct points $u, v$ $of$ $W$, there exists an $\mathbf{S_{gs}}^*$-clopen set $M$containing$u$but not $v$, it follows $W - M$ is $\mathbf{S_{gs}}^*$-open set and$v \in W - M$ , since $M \cap (W - M) = \varphi$.Thus $W$ is $\mathbf{S_{gs}}^*$-$T_2$space.

**Proposition 3.30:** For a space X the following statements are equivalent:

1. X is $\mathbf{S_{gs}}^*$-regular space
2. for each a∈ X and each open set G containing a ,there exist an $\mathbf{S_{gs}}^*$- open sets H containing a such that a∈H⊆ $\mathbf{S_{gs}}^*$cl H ⊆G
3. Each element of X has an neighborhood base consisting of $\mathbf{S_{gs}}^*$-closed sets

**Proof:**

**1→2** Suppose that X is $\mathbf{S_{gs}}^*$-regular space and G is anopen set such that a ∈ G which implies that X-G is closed set and a∉X-G. By Definitions 3.1 there are disjoint $\mathbf{S_{gs}}^*$-open sets H and W such that a ∈ H and X-G ⊆ W ,it follows that a ∈ H⊆X- W ⊆ G ,then a ∈ H⊆ $\mathbf{S_{gs}}^*$cl H⊆ $\mathbf{S_{gs}}^*$cl (X- W) = X- W ⊆ G ,we get

$\mathbf{S_{gs}}^*$cl H ⊆ X- W ⊆ G .Thus a∈H⊆ $\mathbf{S_{gs}}^*$cl H ⊆G

2→**3** Let b∈X, by hypothesis for each open set G, there exist an $\mathbf{S_{gs}}^*$- open set H such that b∈H⊆ $\mathbf{S_{gs}}^*$cl H ⊆G. This gives for each b∈ X the set $\mathbf{S_{gs}}^*$clH from an $\mathbf{S_{gs}}^*$-neighborhood base consisting of $\mathbf{S_{gs}}^*$-closed sets.

3→**1** Suppose that U is an $\mathbf{S_{gs}}^*$-closed set such that a∉U implies that X- U is $\mathbf{S_{gs}}^*$-open set, then it is neighborhood of a. By hypothesis there is an $\mathbf{S_{gs}}^*$- closed set V which contain a and it is an neighborhood of a with V⊆ X- U, then a∈V, U ⊆ X-V=W ,V and U are disjoint $\mathbf{S_{gs}}^*$-open sets .Hence X is $\mathbf{S_{gs}}^*$-regular space

**Proposition 3.31 :** A topological space $W$ is an $\mathbf{S_{gs}}^*$-regular space if and only if for each g∈$W$ and $\mathbf{S_{gs}}^*$-closed set Y such that g∉ Y , there is an $\mathbf{S_{gs}}^*$-open sets P,Q in $W$ such thatg∈ P , Y ⊆ Q and $\mathbf{S_{gs}}^*$cl (P)∩ $\mathbf{S_{gs}}^*$cl(Q) = $\varphi$

**Proof :** Let g∈$W$ and $\mathbf{S_{gs}}^*$- closed set Y such that g∉Y and there is an $\mathbf{S_{gs}}^*$-open sets $M$, N in W such that g∈$M$ , Y ⊆ N and $M \cap N = \varphi$ , that means g∈$M$⊆ $W$ – N⊆ $W$ – Y, it follows that g∈$M$⊆ $\mathbf{S_{gs}}^*$cl ($M$) ⊆$W$ – N⊆ $W$ – Y. Since W is $\mathbf{S_{gs}}^*$-regular space and Proposition3.30 (2) and $M$is an $\mathbf{S_{gs}}^*$-open set, we obtain that there exist an $\mathbf{S_{gs}}^*$-open sets P containing g such that g∈ P ⊆ $\mathbf{S_{gs}}^*$cl (P)⊆$M$. This indicates that Y⊆ N ⊆$W$ – $\mathbf{S_{gs}}^*$cl ($M$) ⊆$W$ –$M$⊆$W$ – $\mathbf{S_{gs}}^*$cl ($P$) and Y ⊆ N ⊆ $\mathbf{S_{gs}}^*$cl ($N$) ⊆$W$ –$M$.
Putting Q =N ,which implies that g∈ P , Y ⊆ Q and $\mathbf{S_{gs}}^*$cl (P)∩$\mathbf{S_{gs}}^*$cl(Q) = $\varphi$

**Proposition 3.32:**The property of a space being $\mathbf{S_{gs}}^*$-regular space is a $\mathbf{S_{gs}}^*$-topological property.

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**Proof:** Suppose that $(W,S)$ be $\mathbf{S_{gs}}^*$-regular space and $f$: $(W,S)\to(\upsilon,\varsigma)$ be homeomorphism. Let $K$ be $\mathbf{S_{gs}}^*$-open set of $\upsilon$ and $g\in K$. By hypothesis $f$ is $\mathbf{S_{gs}}^*$-homeomorphism ,we get there is a unique $e\in W$ such that $g=f(e)$ and $e\in f^{-1}(K)$ where $f^{-1}(K)$ is $\mathbf{S_{gs}}^*$-open subset of $W$ , since $(W,S)$ be $\mathbf{S_{gs}}^*$-regular space ,this gives, there exist an $\mathbf{S_{gs}}^*$- open subset $D$ of $W$ such that $e\in D\subseteq$ $\mathbf{S_{gs}}^*$cl $(D)$ $\subseteq f^{-1}(K)$, it follows that $g\in f(D)\subseteq f(\mathbf{S_{gs}}^*\text{cl }(D))$ = $\mathbf{S_{gs}}^* cl f(D)\subseteq K$, where $f(D)$ is $\mathbf{S_{gs}}^*$-open set of $\upsilon$ .

**<u>Proposition 3.33 :</u>** Let $(W,S)$ be an $\mathbf{S_{gs}}^*$- regular space and $Z$ be $\alpha$-open subspace of $W$,then $Z$ is - $\mathbf{S_{gs}}^*$ regular space.

**Proof:** Suppose that $e\in Z$ where P is closed set in $Z$ such that $e\notin$P, which implies that P= $Z\cap K$ where $K$ is closed set in $W$ and $e\notin K$. Since $(W,S)$ is $\mathbf{S_{gs}}^*$- regular space , that means there are disjoint two $\mathbf{S_{gs}}^*$-open sets H and G in $W$ such that $e\in$ H and $K\subseteq$ G, which implies that $Z\cap H$ and $Z\cap G$ are disjoint $\mathbf{S_{gs}}^*$-open sets in $Z$ containing $e$ and P respectively .

**<u>Proposition 3.34:</u>** If $W$ is strongly $\mathbf{S_{gs}}^*$- normal space, then for each $\mathbf{S_{gs}}^*$-closed set F in $W$ and $\mathbf{S_{gs}}^*$- open set $U$ contains F ,there is an $\mathbf{S_{gs}}^*$-open sets $M$ such that F$\subseteq M\subseteq$ $\mathbf{S_{gs}}^*$cl $M\subseteq U$

**Proof:** Let $U$ be $\mathbf{S_{gs}}^*$-open set contains F, that means $W-U$ and F are disjoint $\mathbf{S_{gs}}^*$-open sets in $W$. By hypothesis $W$ is strongly $\mathbf{S_{gs}}^*$-normal space, this gives, there exist an $\mathbf{S_{gs}}^*$-open sets $M,N$ such that F $\subseteq M$ ,$W-U\subseteq N$ ,$M\cap N=\varphi$.Therefore F $\subseteq$ $M\subseteq$ $\mathbf{S_{gs}}^*$cl $M\subseteq$ $\mathbf{S_{gs}}^*$cl $(W-N)=W-N\subseteq U$ or F $\subseteq M\subseteq$ $\mathbf{S_{gs}}^*$cl$M\subseteq U$

**<u>Proposition 3.35:</u>** A topological space $(W,S)$ is $\mathbf{S_{gs}}^*$-normal space iff for every $\mathbf{S_{gs}}^*$-closed set F and $\mathbf{S_{gs}}^*$-open H containing F, there exist an $\mathbf{S_{gs}}^*$-open sets G such that F$\subseteq$G $\subseteq$ $\mathbf{S_{gs}}^*$clG $\subseteq$ H.

**Proof:** Let $(W,S)$ be $\mathbf{S_{gs}}^*$-normal space and let F$\subseteq$H with F is $\mathbf{S_{gs}}^*$- closed set and H is $\mathbf{S_{gs}}^*$-open, then $W$ -H is $\mathbf{S_{gs}}^*$-closed set and F $\cap W$ -H $=\varphi$. But X is $\mathbf{S_{gs}}^*$-normal space ,hence there exist $\mathbf{S_{gs}}^*$-open sets G,G$^*$ such that F$\subseteq$G , $W$ -H $\subseteq$ G$^*$and G$\cap$G$^*=\varphi$.

Since G$\cap$G$^*=\varphi$ then G$\subseteq$ G$^{*c}$ and $W$ -H$\subseteq$ G$^*\Rightarrow W$ -G$^*$ $\subseteq$H.

Furthermore $W$ -G$^*$ is $\mathbf{S_{gs}}^*$-closed set. Hence F$\subseteq$G $\subseteq$ $\mathbf{S_{gs}}^*$clG$\subseteq W$ -G$^*\subseteq$H .

Conversely, Let $F_1$,$F_2$ be disjoint $\mathbf{S_{gs}}^*$-closed sets ,then $F_1\subseteq W$ -$F_2$ and $W$ -$F_2$ is $\mathbf{S_{gs}}^*$-open set We obtain there exist $\mathbf{S_{gs}}^*$-open sets G such that $F_1\subseteq$G$\subseteq$ $\mathbf{S_{gs}}^*$clG$\subseteq W$ -$F_2$. But $S_{gs}$¿clG$\subseteq W$ -$F_2$ ,we get $F_2\subseteq$ $\mathbf{S_{gs}}^*$cl($W$ –G)and G$\subseteq$ $\mathbf{S_{gs}}^*$clG $\Rightarrow$ G$\cap\mathbf{S_{gs}}^*$cl($W$ -H) $=\varphi$. Furthermore $\mathbf{S_{gs}}^*$cl($W$ -H) is $\mathbf{S_{gs}}^*$-open set. Thus $F_1\subseteq$G , $F_2\subseteq$ $\mathbf{S_{gs}}^*$cl($W$ -H) ,with G , $\mathbf{S_{gs}}^*$cl($W$ -H) are disjoint $\mathbf{S_{gs}}^*$-open sets. Hence $W$ is $\mathbf{S_{gs}}^*$-normal space .

**<u>Proposition 3.36 :</u>** Every $T_1$–space, $\mathbf{S_{gs}}^*$-normal space is $\mathbf{S_{gs}}^*$-regular space .

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**Proof:** Suppose that $(W,S)$ be $\mathbf{S_{gs}}^*$-normal space, $e\in W$, $K$ is closed subset in $W$ such that $e\notin K$, since $W$ is $T_1$ -space , implies that $\{e\}$ is closed subset in $W$ with $\{e\}\cap K=\varphi$, also since $W$ is $\mathbf{S_{gs}}^*$-normal space, then there exist two disjoint $\mathbf{S_{gs}}^*$-open sets $M,N$ such that $\{e\}\subseteq M$ , that means $e\in M, K\subseteq N, M\cap N=\varphi$ .Thus $(W,S)$ is $\mathbf{S_{gs}}^*$-regular space

**Proposition 3.37:** Every strongly $\mathbf{S_{gs}}^*$-normal space ,$T_1$ -space , is $\mathbf{S_{gs}}^*$-regular space
**Proof:** Let $K$ is $\mathbf{S_{gs}}^*$-closed subset in strongly $\mathbf{S_{gs}}^*$-normal space $W$ and $e\in W$ such that $e\notin K$, since $W$ is $\mathbf{S_{gs}}^*$-$T_1$ -space. By Proposition 3.9, each $\{e\}$ is closed subset in $W$ also , since $W$ is $\mathbf{S_{gs}}^*$-normal space, then there exist two disjoint $\mathbf{S_{gs}}^*$-open sets $M,N$ such that $\{e\}\subseteq M, K\subseteq N, M\cap N=\varphi$, so $e\in M, K\subseteq N$ , $M\cap N=\varphi$. Hence $(W,S)$ is $\mathbf{S_{gs}}^*$-regular space.

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