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An Orthogonal Higher Reverse Left (resp. Right) Centralizer on Semiprime Rings

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Abstract

Let R be a semiprime ring ,we prove the following main result :
Let R be a 2-torsion free semiprime ring , $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ be two higher reverse left (resp.right) centralizers of R .Then t_n and h_n are orthogonal if and only if $t_n(x)\, h_n(y) + h_n(x)\, t_n(y) = 0$, for all x , y $\in$ R and n $\in$N .

Key Words : semiprime ring , higher reverse left (resp. right)centralizer , orthogonal higher reverse left (resp. right) centralizers .
Mathematic Subject classification : 16N60 , 42C05 , 33C45 .

I - Introduction:

A ring R is called semiprime if xRx = (0) implies x = 0, such that x $\in$ R [3] .
Let R be a ring then R is called 2-torsion free if 2x = 0 implies x = 0, for all x $\in$ R [3] .
A left (resp. right) centralizer of a ring R is an additive mapping t: R $\longrightarrow$ R which satisfies the equation : t (xy) = t(x) y (resp.t (xy) = x t (y)) , for all x ,y $\in$ R . t is called a centralizer of R if it is both a left and a right centralizer [5] . A left (resp. right) Jordan centralizer of a ring R is an additive mapping t: R $\longrightarrow$ R which satisfies the equation : t (x^2) = t(x) x (resp. t (x^2) = x t(x)) , for all x $\in$ R . t is called a Jordan centralizer of R if it is both a left and a right Jordan centralizer [5] . Jarullah and Salih in [4] introduced the concepts of a higher reverse left (resp. right) centralizer and a Jordan higher reverse left (resp. right) centralizer on rings as follows : Let $t = (t_i)_{i\in N}$ be a family of additive mappings of a ring R into itself .Then t is called a higher reverse left (resp. right) centralizer of R if for all x , y $\in$ R and n $\in$ N

$$t_n(xy)=\sum_{i=1}^{n} t_i(y)\, t_{i-1}(x) \quad (\text{resp. } t_n(xy)=\sum_{i=1}^{n} t_{i-1}(y)\, t_i(x)) \ .$$

Let $t = (t_i)_{i\in N}$ be a family of additive mappings of a ring R into itself .Then t is called a Jordan higher reverse left (resp. right) centralizer of R , if the following equation holds :

$$t_n(x^2)=\sum_{i=1}^{n} t_i(x)\, t_{i-1}(x) \quad (\text{resp. } t_n(x^2)=\sum_{i=1}^{n} t_{i-1}(x)\, t_i(x)) \ , \text{ for all } x \in R \text{ and } n \in N \ .$$

In this paper , we define and study the concept of orthogonal higher reverse left (resp.right) centralizers of semiprime rings and we prove some of lemmas and theorems about orthogonally one of these theorems is :
Let R be a 2-torsion free semiprime ring , $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ be two higher reverse left (resp. right) centralizers of R, suppose that $t^2_n = h^2_n$ .Then $t_n + h_n$ and $t_n - h_n$ are orthogonal .
In our work we need the following Lemmas :

Lemma (1.1): [2]
Let R be a 2-torsion free semiprime ring and x, y be elements of R, then the following conditions are equivalent :

(i) $xry = 0$, for all $r \in R$
(ii) $yrx = 0$, for all $r \in R$
(iii) $xry + yrx = 0$, for all $r \in R$
If one of these conditions is fulfilled ,then $xy = yx = 0$.

Lemma (1.2): [1]

Let R be a 2-torsion free semiprime ring and x , y be elements of R if $xry + yrx = 0$, for all $r \in R$,then $xry = yrx = 0$.

II - Orthogonal Higher Reverse Left (resp. Right) Centralizers on Semiprime Rings :

In this section we will introduce the concept of orthogonal higher reverse left (resp. right) centralizers on semiprime rings.

Definition (2.1):

Two higher reverse left (resp.right)centralizers $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ of a ring R are called orthogonal if

$t_n(x)\, R\, h_n(y) = (0) = h_n(y)\, R\, t_n(x)$, for all $x , y \in R$ and $n \in N$. Where

$$t_n(x)\, R\, h_n(y) = \sum_{i=1}^{n} t_i(x)\, z\, h_i(y) \text{ , for all } z \in R$$

Lemma (2.2):

Let R be a semiprime ring ,suppose that $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ are two higher reverse left (resp.right) centralizers of R , satisfy $t_n(x)\, R\, h_n(x) = (0)$, for all $x \in R$ and $n \in N$. Then $t_n(x)\, R\, h_n(y) = (0)$, for all $x , y \in R$ and $n \in N$.

Proof :

Suppose that $t_n(x)\, R\, h_n(x) = \sum_{i=1}^{n} t_i(x)\, z\, h_i(x) = 0$, for all $x , z \in R$ and $n \in N$ ...(1)

Replace x by $x + y$ in (1) , we have that

$$\sum_{i=1}^{n} t_i(x + y)\, z\, h_i(x + y) = 0$$

$$\sum_{i=1}^{n} t_i(x)\, z\, h_i(x) + t_i(x)\, z\, h_i(y) + t_i(y)\, z\, h_i(x) + t_i(y)\, z\, h_i(y) = 0$$

Therefore , by our assumption and Lemma (1.1) , we get

$$\sum_{i=1}^{n} t_i(x)\, z\, h_i(x) = 0 \text{ , , for all } x , y , z \in R$$

Thus , $t_n(x)\, R\, h_n(y) = (0)$, for all $x , y \in R$ and $n \in N$.

Lemma (2.3):

Let R be a 2-torsion free semiprime ring , $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ be two higher reverse left (resp.right) centralizers of R .Then t_n and h_n are orthogonal if and only if $t_n(x)\, h_n(y) + h_n(x)\, t_n(y) = 0$, for all $x , y \in R$ and $n \in N$.

Proof :

Suppose that t_n and h_n are orthogonal T.P.

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$t_n(x)\, h_n(y) + h_n(x)\, t_n(y) = 0$, for all $x , y \in R$ and $n \in N$ Since t_n and h_n are orthogonal, we have that

$$\sum_{i=1}^{n} t_i(x)\, z\, h_i(y) = 0 = \sum_{i=1}^{n} h_i(y)\, z\, t_i(x)$$

Therefore , by Lemma (1.1) , we get the require result
Conversely , it's clear by using Lemma (1.2)

<u>Theorem (2.4):</u>
Let R be a 2-torsion free semiprime ring , $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ be two higher reverse left (resp.right) centralizers of R , where t_n and h_n are commuting .Then the following conditions are equivalent, for all $n \in N$:
(i) t_n and h_n are orthogonal
(ii) $t_nh_n = 0$
(iii) $h_nt_n = 0$
(iv) $t_nh_n + h_nt_n = 0$

<u>Proof :</u> **(i)** $\Leftrightarrow$ **(ii)**
Suppose that t_n and h_n are orthogonal
T.P. $t_nh_n = 0$
Since t_n and h_n are orthogonal, we have that

$$\sum_{i=1}^{n} t_i(x)\, z\, h_i(y) = 0$$

$$\sum_{i=1}^{n} t_i\,(t_i(x)\, z\, h_i(y)) = 0$$

$$\sum_{i=1}^{n} t_i(h_i(y))\; t_{i-1}(z)\; t_{i-1}(t_i(x)) = 0$$

By Lemma (1.1) , we have that

$$\sum_{i=1}^{n} t_i(h_i(y))\; t_{i-1}(t_i(x)) = 0$$

Right multiply by $t_i(h_i(y))$, we have that

$$\sum_{i=1}^{n} t_i(h_i(y))\; t_{i-1}(t_i(x))\; t_i(h_i(y)) = 0$$

Since R is a semiprime ring , we have that

$$\sum_{i=1}^{n} t_i(h_i(y)) = 0 \text{, for all } y \in R \;\Rightarrow\; t_nh_n = 0 \text{ , for all } n \in N$$

Conversely , suppose that $t_nh_n = 0$, for all $n \in N$
T.P. t_n and h_n are orthogonal
$t_n(h_n(xy)) = 0$

$$\sum_{i=1}^{n} t_i\,(h_i(y)\, h_{i-1}(x)) = 0$$

$$\sum_{i=1}^{n} t_i\,(h_{i-1}(x))\; t_{i-1}(h_i(y)) = 0$$

Replace $h_{i-1}(x)$ by x , we have that $\sum_{i=1}^{n} t_i(x)\, t_{i-1}(h_i(y)) = 0$

Right multiply by $h_i(y)$, we have that

$$\sum_{i=1}^{n} t_i(x)\, t_{i-1}(h_i(y))\, h_i(y) = 0 \text{ , for all } x , y \in R \qquad \ldots(1)$$

Since t_n and h_n are commuting , we have that

$$\sum_{i=1}^{n} h_i(y)\, t_{i-1}(h_i(y))\, t_i(x) = 0 \text{ , for all } x , y \in R \qquad \ldots(2)$$

By (1) and (2) , we get t_n and h_n are orthogonal .

Proof : **(i) ⇔ (iii)**
By the same way in (i) ⇔ (ii) , we get (i) ⇔ (iii) .

Proof : **(i) ⇔ (iv)**
Suppose that t_n and h_n are orthogonal
T.P. $t_nh_n + h_nt_n = 0$, for all $n \in N$
By (ii) and (iii) , we get the require result
Conversely , suppose that $t_nh_n + h_nt_n = 0$, for all $n \in N$
T.P. t_n and h_n are orthogonal
$(t_nh_n + h_nt_n)(xy) = 0$

$$\sum_{i=1}^{n} t_i(h_i(xy)) + h_i(t_i(xy)) = 0 \text{ , for all } x , y \in R$$

$$\sum_{i=1}^{n} t_i(h_i(y)\, h_{i-1}(x)) + h_i(t_i(y)\, t_{i-1}(x)) = 0 \text{ , for all } x , y \in R$$

$$\sum_{i=1}^{n} t_i(h_{i-1}(x))\, t_{i-1}(h_i(y)) + h_i(t_{i-1}(x))\, h_{i-1}(t_i(y)) = 0 \text{ , for all } x , y \in R$$

Replace $h_{i-1}(x)$ by x and $t_{i-1}(x)$ by x , we have that

$$\sum_{i=1}^{n} t_i(x)\, t_{i-1}(h_i(y)) + h_i(x)\, h_{i-1}(t_i(y)) = 0$$

Replace $t_{i-1}(h_i(y))$ by $h_i(y)$ and $h_{i-1}(t_i(y))$ by $t_i(y)$, we have that

$$\sum_{i=1}^{n} t_i(x)\, h_i(y) + h_i(x)\, t_i(y) = 0$$

Thus , $t_n(x)\, h_n(y) + h_n(x)\, t_n(y) = 0$, for all $x , y \in R$ and $n \in N$.
By Lemma (2.3) , we get the require result .

Theorem(2.5):
Let R be a 2-torsion free semiprime ring , $t=(t_i)_{i\in N}$ and $h=(h_i)_{i\in N}$ be two higher reverse left (resp.right) centralizers of R , suppose that $t^2_n = h^2_n$. Then $t_n + h_n$ and $t_n - h_n$ are orthogonal .

Proof :
$((t_n + h_n)(t_n - h_n) + (t_n - h_n)(t_n + h_n))(x)$

$$\sum_{i=1}^{n} t^2_i (x) - t_i(x)\, h_i(x) + h_i(x)t_i(x) - h^2_i (x) + t^2_i (x) + t_i(x)\, h_i(x) - h_i(x)t_i(x) - h^2_i (x) = 0$$

Therefore , $((t_n + h_n) (t_n - h_n) + (t_n - h_n) (t_n + h_n))(x) = 0$
By Theorem (2.4)(iv) $\Rightarrow$ (i) , we get the require result .

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