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Estimate of Fractal Dimensions in the Ricker Nonlinear Population Model

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Abstract: This paper is concerned with suitable formulation in order to estimate box-counting dimension, correlation dimension and information dimension in nonlinear discrete systems. We consider the Ricker nonlinear population model: $f(x) = x\, e^{r(1-\frac{x}{k})}$, where r is the control parameter and k is the carrying capacity, and our method reveals that the values of these dimensions are respectively 0.531004…., 0.506938… and 0.749124…... Our method can be extended to higher dimensions for estimate of various fractal dimensions.

Key Words: Nonlinear model / Dimension / Carrying capacity / Discrete system

2010 AMS Subject Classification: 37 G 15, 37 G 35, 37 C 45

1. Introduction:

The study of various fractal dimensions is an emerging research topic in nonlinear dynamical systems. The dimensionality of an attractor gives us an estimate of the number of active degrees of freedom for the concerned system and the geometric objects with dimensionalities that are not integers play a fundamental role in the dynamics of chaotic systems .

One answer for quantifying chaos lies in a desire to be able to specify quantitatively whether or not a system's apparently erratic behavior is indeed chaotic. We would like to have some definite, quantitative way of recognizing chaos and sorting out "true" chaos from just noisy behavior or erratic behavior due to complexity, (that is, due to a large number of degrees of freedom). Secondly, some of these quantifiers can give us an estimate of the number of active degrees of freedom for the system. A third reason for quantifying chaotic behavior is that we might anticipate, based on our previous results with the universality of the scenarios connecting regular behavior to chaotic behavior, that there are analogous universal features, both qualitative and quantitative, that describe a system's behavior and changes of its behavior within its chaotic regime as parameters of the system are changed . Another category of quantifiers focuses on the geometric aspects of the attractors. In practice, we let the trajectories run for a long time and collect a long time series of data. Now, the geometric question is about how this series of points is distributed in state space, and this geometry provides important clues about the nature of the trajectory dynamics. Different dimensions show up in yet another aspect of nonlinear dynamics. As we know, many nonlinear systems show sensitivity to initial conditions in the sense that trajectories that are initially nearby in state space may evolve, for dissipative systems, to very different attractors. In some cases, the attractors may be chaotic attractors. As we also know, the set of initial conditions that gives rise to trajectories ending on a particular attractor constitutes the basin of attractor for that attractor. For many nonlinear systems, the boundaries of these basins of attractors are rather complex geometric objects, best characterized with fractal dimensions. We now evolve some suitable formulae in order to estimate various desired dimensions [1,2,4,7,8]

2. The Main Results:

2.1 Box-counting dimension, [4,5,8,12] :

Let F be any non empty bounded subset of R^n and let $N_\delta(F)$ be the smallest number of sets (or boxes or cells) of diameter at most δ which can cover F. The lower and upper box counting dimensions of F respectively are given by

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$ and

$$\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

If these are equal, we refer to the common value as the box counting dimension (or simply Box-dimension) of F. Then we write

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

We assume that $\delta > 0$ is sufficiently small to ensure that - $\log \delta$ and similar quantities are strictly positive. To avoid problems with 'log 0' and 'log ∞', we generally consider box dimension only for non-empty bounded sets

2.1.1 The method for Calculation of box-counting dimension

Let us develop a suitable mechanism for evaluating box dimension of our population model. The Box-counting dimension is also conveniently defined by the formula,

$$D = \lim_{s \to 0} \log[N(s)] / \log[1/s],$$

where N(s) is the number of boxes covering the attractor with the scaling s (that is , "s" represents size of the box) when the number of iteration tends to infinity and the scaling 's' tends to zero. For calculation purpose we first of all find the attracting region and then divide that attracting region with some suitable scaling's'. Then we iterate the relation for some fixed number of iterations and count the number of boxes. We calculate the number of boxes for different scaling and put it in log-log graph of N(s) versus (1/s). If it reveals a straight line then the slope of that straight line will give the required dimension. In computation the number of boxes N(s) we mean the total number of boxes visited by infinite number of iterations. However in practical process we can iterate up to some finite number of iterations. So by computation we may never get the exact value of N(s).

When repeating the same procedure using smaller sizes s we expect to find that the count N(s) scales like a power of s, $N(s) \propto s^{-D_b}$, where D_b is the box-counting dimension . The attractor is usually computed using a great deal iterations of the model. After each such iteration which is needed to get sufficiently close to the attractor, we check if the current point is in a box that we have not yet visited in which case we increase our count by 1. After we have visited all boxes that cover the attractor we stop the iteration, repeat the whole procedure for a different size s and finally compute D_b from the power law as the slope of the graph of log (N(s)) versus log (1/s).

However, it is not so easy to count N(s) directly, all we can expect is a count N(s,n) which depends on the number n of iterations performed. [Here, N(s, n) is the count of boxes of linear size s that contain one or more iterates from an orbit on our model computed for a length of n]. Given a table of values of N(s,n) if we can find a relation, then we can extrapolate from our count N (s , n) to arrive at an estimate for $N(s) = \lim_{n \to \infty} [N(s,n)]$, which is needed for the dimension calculation . These issues were addressed in 1983 in a paper by Peter Grassberger [5]. His tabulated data suggested a behavior

$$N(s, n) \approx N(s) - \text{const. } s^{-a} n^{-b} \qquad (1.1)$$

for large number n of iterations. Throughout our calculation ,we take $s = 2^{-8}, 2^{-9}, \ldots 2^{-20}$, as the size of the boxes . The calculation is done at the accumulation point, r = 2.69236885439051

The program is run for 4000000 iterations with 3 different initial values of $x = 10^{m-6}+0.01$ where m=1,2,3 .For each initial value the N(n, s) is calculated and then taken the average. Thus N(n, s) used in our calculation is the average of three N(n, s) calculated for three initial values. If equation (1.1) is well fitted in our case then

$N(n_1,s) – N(n_2,s)= N(s) – \text{const. } s^{-a} .n_1^{-b} - N(s) + \text{const. } s^{-a} .n_2^{-b}.$

$= \text{const.}s^{-a} (n_2^{-b}- n_1^{-b}).$

If n_1 is large enough than n_2, we can neglect n_1^{-b} with respect to n_2^{-b}.

So, we can say

$$N(n_1,s) – N(n_2,s) \approx \text{const.}s^{-a} (n_2^{-b}). \quad (1.2)$$

If we fix n_2 in (2) then $\log(N(n_1,s) – N(n_2,s))$ vs. $\log(n_1)$,where n_1 is very large compared to n_2 , be a straight line parallel to the abscissa.

We vary n_1 from 3000000 to 4000000 with an increment of 100 .We set n_2 as 100 and $s=2^{-20}$, $\log(N(n_1, s) – N(100, s))$ as Y-axis and $\log(n_1)$ as X-axis, then $\log(N(n_1, s) – N(100, s))$ vs. $\log(n_1)$ graph is as follows:

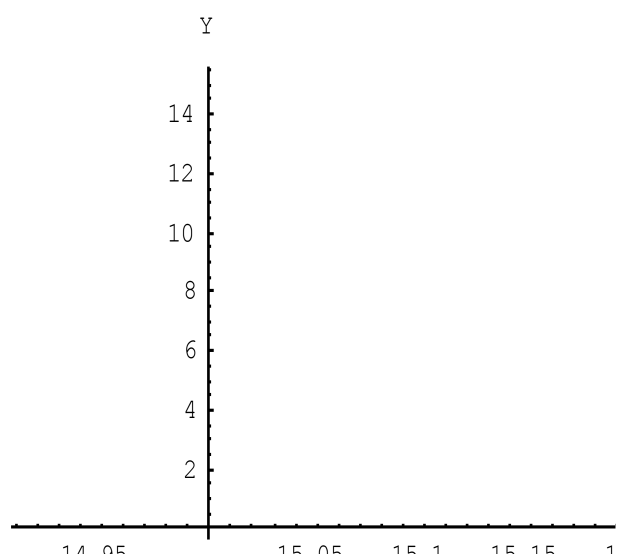

Fig 1: Graph of $\log(N(n_1,s) – N(100,s))$ as Y-axis and $\log(n_1)$ as X-axis

Again if we take n1 large enough compared to n_2 but fix both of them and vary "s" then we have

$N(n_1, s) – N(n_2, s)= N(s) – \text{const.}n_1^{-b} s^{-a} - N(s) + \text{const.}n_2^{-b} s^{-a}$

$$\approx \text{const.}s^{-a}(n_2^{-b}) \quad (1.3)$$

If we plot $\log(N(n_1,s) – N(n_2,s))$ vs.$\log(1/s)$ then we will have a straight line whose slope should be "a".

We have taken log(N(4000000,s) – N(100,s)) vs log(1/s) i.e. log(1/s) in the X-axis and log(N(4000000,s) – N(100, s)) in the Y-axis and $s=2^{-8},2^{-9},\ldots 2^{-20}$. The graph is as follows:

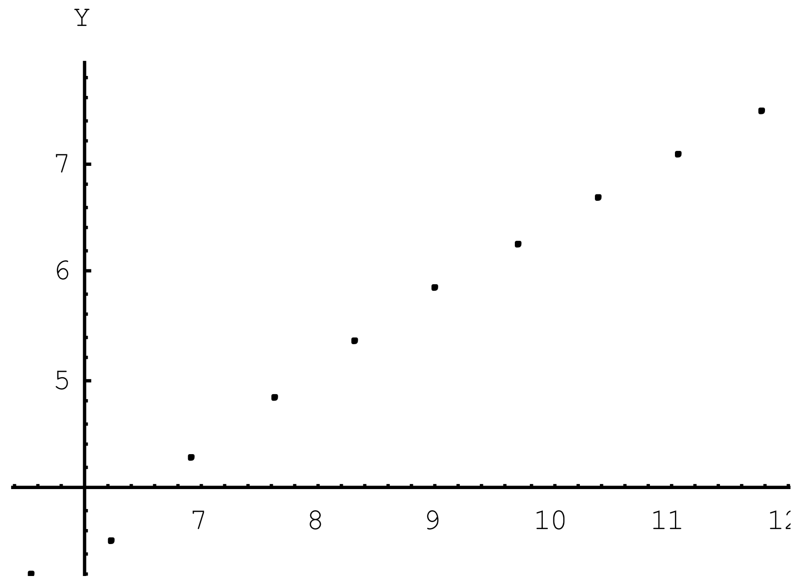

Fig 2: log(1/s) in the X-axis and log(N(4000000,s) – N(100,s)) in the Y-axis

From the graph we can see that the plotted points more or less follow a straight line path .So, we can now say that equation (1.1) can be used in our case also. The straight line when fitted by least square method gives the slope i.e. the value of "a" as 0.597345. Again, from equation (1.3) putting n_1=4000000 and n_2=100 we have

N(4000000,s) – N(100,s) ≈ const.$s^{-a}(100^{-b})$ [i.e. we have neglected 4000000^{-b}]

Again putting n_1=4000000 and n_2=200 we have

N(4000000,s) – N(200,s) ≈ const.$s^{-a}(200^{-b})$

$$\frac{N(4000000,s) - N(100,s)}{N(4000000,s) - N(200,s)} = \left(\frac{1}{2}\right)^{-b}$$

Therefore $\frac{log\left(\frac{N(4000000,s) - N(100,s)}{N(4000000,s) - N(200,s)}\right)}{\log(2)}$= b, which should be equal for all values of s, but in real data the value may be a little bit different. So we have taken

b =$\frac{\sum_s \frac{log\left(\frac{N(4000000,s) - N(100,s)}{N(4000000,s) - N(200,s)}\right)}{\log(2)}}{13}$,where s = $2^{-8},2^{-9},\ldots.2^{-20}$ (13 values of "s").

The value of "b", we have got is 0.4572. Next, we calculate the value of the constant . From (1.1) we have

$N(200,2^{-10}) – N(100,2^{-10})$=const.$(2^{-10})^{-a}(100^{-b}- 200^{-b})$.

Therefore, const = $\frac{N(200,2^{-10}) - N(100,2^{-10})}{2^{10a}(100^{-b} - 200^{-b})}$

The value of the constant is 14.9157.

Now we can calculate N(s)=N(n,s)+ const.s^{-a} n^{-b} , for different values of s, n .The following table represents the values of N(n,s)+ const. s^{-a} n^{-b}, first column for 1/s =2^8, next columns for 1/s =$2^9,2^{10},\ldots.,2^{20}$. The first row for n=1000000 , next n=1100000,1200000,…….,4000000. Clearly in a particular column the values are almost same which verifies that it is independent of n and thus represents N(s).

Table: 1

[Calculation of $N(s)=N(n,s)+ const.s^{-a} n^{-b}$, for different values of s & n]

86.4063	96.119	142.693	204.895	291.875	427.196	612.87	890.752	1293.97	1894.38	2
86.3747	96.0712	142.621	204.785	291.71	426.946	612.491	890.18	1293.1	1893.07	2
86.3471	96.0295	142.557	204.69	291.565	426.727	612.16	889.679	1292.35	1891.93	2
86.3227	95.9925	142.502	204.605	291.437	426.533	611.867	889.236	1291.67	1890.91	2
86.3008	95.9594	142.452	204.529	291.322	426.36	611.605	888.839	1291.07	1890.	2
86.2811	95.9296	142.406	204.461	291.219	426.204	611.369	888.482	1290.53	1889.18	2
86.2632	95.9026	142.366	204.399	291.126	426.062	611.155	888.158	1290.04	1888.44	2
86.2469	95.8779	142.328	204.343	291.04	425.933	610.959	887.862	1289.6	1887.77	2
86.232	95.8553	142.294	204.291	290.962	425.814	610.779	887.59	1289.18	1887.14	2
86.2182	95.8344	142.262	204.243	290.889	425.705	610.614	887.34	1288.81	1886.57	2
86.2054	95.815	142.233	204.199	290.823	425.604	610.461	887.108	1288.45	1886.04	2
86.1935	95.7971	142.206	204.158	290.76	425.509	610.318	886.892	1288.13	1885.55	2
86.1824	95.7803	142.181	204.119	290.702	425.422	610.185	886.691	1287.82	1885.09	2
86.172	95.7646	142.157	204.083	290.648	425.339	610.061	886.503	1287.54	1884.66	2
86.1623	95.7499	142.134	204.05	290.597	425.262	609.944	886.326	1287.27	1884.25	2
86.1531	95.736	142.114	204.018	290.549	425.189	609.834	886.16	1287.02	1883.87	2
86.1445	95.7229	142.094	203.988	290.503	425.121	609.73	886.003	1286.78	1883.51	2
86.1363	95.7105	142.075	203.96	290.461	425.056	609.632	885.855	1286.56	1883.17	2
86.1286	95.6988	142.057	203.933	290.42	424.995	609.539	885.714	1286.35	1882.85	2
86.1212	95.6877	142.04	203.907	290.382	424.936	609.451	885.581	1286.14	1882.54	2
86.1142	95.6771	142.024	203.883	290.345	424.881	609.367	885.454	1285.95	1882.25	2
86.1076	95.6671	142.009	203.86	290.31	424.828	609.288	885.333	1285.77	1881.98	2
86.1012	95.6574	141.995	203.838	290.277	424.778	609.211	885.218	1285.6	1881.71	2
86.0951	95.6483	141.981	203.817	290.245	424.73	609.139	885.108	1285.43	1881.46	2
86.0893	95.6395	141.967	203.797	290.214	424.684	609.069	885.002	1285.27	1881.22	2
86.0838	95.631	141.955	203.778	290.185	424.64	609.002	884.901	1285.12	1880.99	2
86.0784	95.623	141.943	203.759	290.157	424.597	608.938	884.804	1284.97	1880.77	2
86.0733	95.6152	141.931	203.742	290.131	424.557	608.877	884.711	1284.83	1880.55	2
86.0684	95.6078	141.92	203.724	290.105	424.518	608.818	884.622	1284.69	1880.35	2
86.0636	95.6006	141.909	203.708	290.08	424.48	608.761	884.536	1284.56	1880.15	2
86.0591	95.5937	141.898	203.692	290.056	424.444	608.706	884.453	1284.44	1879.96	2

2.1.2 Slope of the graph which gives box dimension

The graph of log(N(s)) vs.log(1/s) is shown bellow. log(N(s)) is taken along the X-axis and log(1/ s) is taken along the Y-axis. The value of $1/s = 2^9, 2^{10}, \ldots\ldots, 2^{18}$.

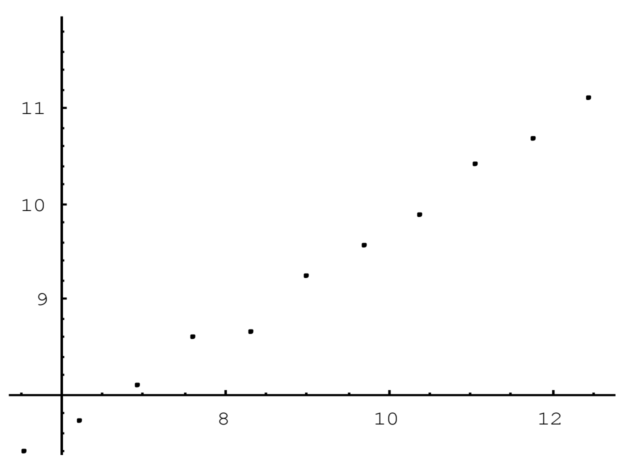

Fig 3 log(N(s)) along the X-axis and log(1/ s) along the Y-axis.

The slope of the line fitted with the help of Least Square method is 0.531004 with a mean deviation of 0.0218148.

Hence the Box- counting dimension is, $D_b = 0.531004\ldots$

2.2 Correlation dimension [2,3,4,9,10]

Grassberger and Procaccia[5,6] introduced a dimension based on the behavior of a so- called correlation sum, (correlation integral). This dimension is called the Correlation Dimension D_c and has been widely used to characterize chaotic attractors. To define the correlation dimension, we first let a trajectory (on an attractor) evolve for a long time , and we collect as data the values of N trajectory points. Then for each point i on the trajectory, we ask for the relative number of trajectory points lying within the distance R of the point i excluding the point i itself. Call this number as Ni (R). Next, we define pi (R) to be the relative number of points within the distance R of the i th points: $p_i (R) = N_i / (N - 1)$.

[We divide by N – 1 because there are at most N – 1 other points in the neighborhood besides the point i] Finally, we compute the correlation sum:

$$C(R) = 1/N \sum_{i=1}^{N} p_i(R) \tag{1.4}$$

Here, C (R) is defined such that C(R) = 1 if all the data points fall within the distance R of each other . If R is smaller than the smallest distance between trajectory points, then $p_i = 0$ for all i, and C (R) = 0 . The relative number p_i itself can be written in more formal terms by introducing the **Heaviside step function** Θ :

$\Theta(x) = 0$ if $x < 0$, $\Theta(x) = 1$ if $x \geq 0$

Using this function, we can write

$$p_i(R) = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \Theta(R - |x_i - x_j|) \tag{1.5}$$

In Equation (1.5), the Heaviside function contributes 1 to the sum for each x_j within the distance R of the point x_i (excluding j = i) ; otherwise, it contributes 0. In terms of the Heaviside function the correlation sum can be written

$$C(R) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \Theta\left(R - |x_i - x_j|\right) \tag{1.6}$$

Often, the limit $N \to \infty$ is added to assure that we characterize the entire attractor. The correlation dimension D_c is then defined to be the number that satisfies

$$C(R) = \lim_{R\to 0} kR^{D_c} \tag{1.7}$$

Or after taking logarithms

$$D_c = \lim_{\mathbf{R}\to 0} \frac{\log \mathbf{C}(\mathbf{R})}{\log \mathbf{R}} \tag{1.8}$$

For convenience of interpretation, we use logarithms to the base 10, though some other workers prefer to use base 2 logarithms.

$D_c \log R = \log(C(R))$ as R tends to 0 (1.9)

From (1.9), we can see that if log-log graph of R vs. C(R) gives a straight line then the slope gives the correlation dimension as R tends to 0.

One obvious difficulty is here that in practical point of view it is difficult to take R tending to zero. As we can see that the graph is not a straight line over a large range of data, we have to find the region of R where it maintains the scaling law i.e. where the graph is a straight line, which is called the scaling region. The slope of that region gives D_c i.e. the correlation dimension and we need not take $R \to 0$.

Following the theory we have taken judiciously 56 different values of R and correspondingly, 56 different values of C(R) in the scaling region with the help of a computer program using the formula of C(R).Then we have plotted the points in the graph taking log(C(R)) in the y-axis and log(R) in the x-axis. We have seen that all the points lies almost in a straight line the slope of which in the scaling region gives the correlation dimension D_c as shown below: Our Computer provides the following points in order to draw the required straight line .

Table 2

[Points in the scaling region for Correlation Dimension]

{-12.203701,-4.18129455570234}, {-11.973442,.04803221948295}, {-11.743184,-3.91584811798623},{-11.512925,-3.78686760077528}, {-11.282667,-3.65821155870166},{-11.052408,-3.53229910167620},-10.822150,-3.40688900035910}, {-10.591891,-3.28285552847134}, {-10.361633,-3.15975343635797},{-10.131374,-3.03513939986868}, {-9.901116,-2.91087091256956}, {-9.670857,-2.78902834746367},{-9.440599,-2.66606518453294}, {-9.210340,-2.54621159161840}, {-8.980082,-2.42702186623428},{-8.749823,-2.30927626581726}, {-8.519565,-2.19161686168172}, {-8.289306,-2.07120417720775},{-8.059048,-1.95095302266108}, {-7.828789,-1.83241851616655}, {-7.598531,-1.71170903152327},{-7.368272,-1.59521932296713}, {-7.138014,-1.48052771111899}, {-6.907755,-1.36647722166283},{-6.677497,-1.25265353697497}, {-6.447238,-1.13540936302150},{ -6.216980,-1.01647265523083},{-5.986721,-0.89913912015864}, {-5.756463,-0.77750073168521} ,{-5.526204,-0.66256207861259},{-5.295946,-0.55067879522417}, {-5.065687,-0.43998807728133}, {-4.835429,-0.33063111849391},{-4.605170,-0.21555286315557}, {-4.374912,-0.09719636578439}, {-.144653,0.01916291152964},{-3.914395,0.14346265087161}, {-3.684136,0.26080403674555}, {-3.453878,0.37179844692993},{-3.223619,0.47899233865072}, {-2.993361,0.58243343009505}, {-2.763102,0.68836225289071},-2.532844,0.79468841299204}, {-2.302585,0.91058595579679}, {-2.072327,1.02663256740167},{-1.842068,1.13635155246955}, {-1.611810,1.25716889993497}, {-1.381551,1.37419218393186},{-1.151293,1.48985052147819}, {-0.921034,1.59663881643950}, {-0.690776,1.70254494662926},{-0.460517,1.82036547410696}, {-0.230259,1.94765077793873}, {0.000000,2.08096475757232}

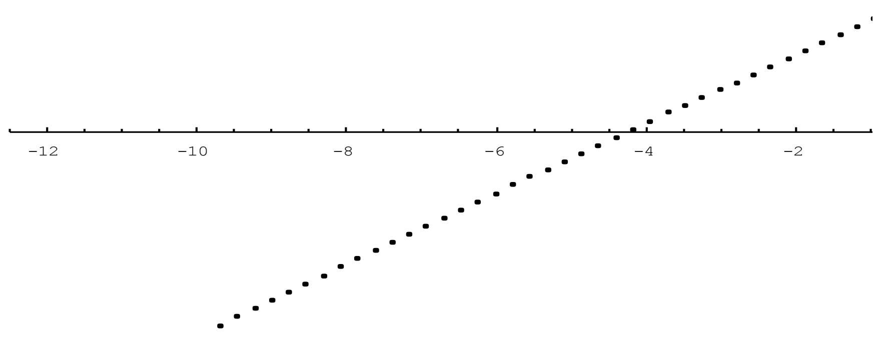

Fig 4 : Graph of log(R) - log (C (R)) and the slope determines the correlation dimension.

By applying the statistical methods as mentioned above, we have found that

the coefficient of correlation, r_{cc} = 0.999839895096. Again, by applying the least square method, we obtain the regression line as y= a + bx, where a= 3.427190895185 and b = 0.5069380996583872. So, the slope of the above straight line is almost 0.5069380996583872. which is regarded as the correlation dimension with a mean deviation of 0.0242335635635499526 , [mean deviation can be used to provide an estimate of the uncertainty to be associated with the average value] . Hence the Correlation Dimension is D_c = 0.5069380996583872…

2.3 Information dimension [2,3,6,8,11] :

The calculation of box-counting dimension is very elaborate in case of even simple one dimension. So we have to put larger effort in case of higher dimensions, say , 2,3,4 dimensions. Further, box counting dimension ignores how many points entered in one box. That is, we are not counting the weightage of the boxes. That is why a different dimension is necessary. Information dimension removes this to some extent.The information dimension can be calculated by the following formula:

$$\mu(B) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} I_B(\chi_k)$$

Let, where $I_B(\chi_k) = 1,\ if\ \ \chi_k \in B\ \ and\ = 0$, otherwise.

Thus $\sum_{k=0}^{n} I_B(\chi_k)$ is the number of points from the finite orbit x_0 , x_1 ,x_2,……. which fall in the set B .

Let us consider partitioning the set F into boxes B_i of size δ .

Let $I(\delta) = \sum_i \mu(B_i)\log_2\left(\frac{1}{\mu(B_i)}\right)$,

then $D_I(F) = \lim_{\delta\to 0} \frac{I(\delta)}{\log_2 \delta}$

By the formula we can set a computer algorithm to find the value. But just like the box counting dimension, it is difficult to get the sum up to N(s). Further the process we have added (by extrapolating) we can get the value. Moreover, to get the probability for each extrapolated box is very difficult. So in order to avoid the problem to some extent ,we have taken the no of iteration up to 100000000.

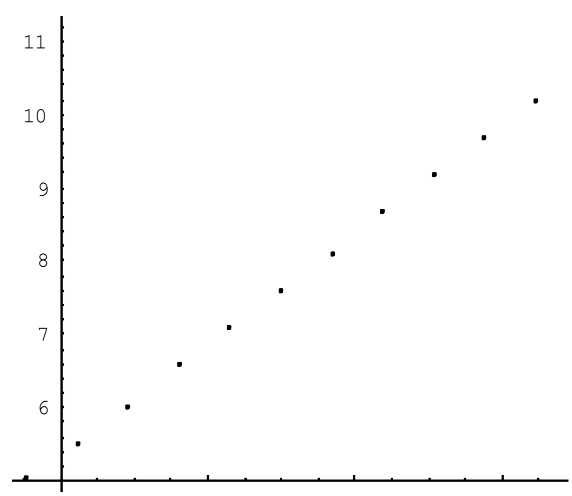

Fig 5 I (δ) along the x- axis and log$_2$ (δ) along y- axis

The information dimension at the control parameter 2.69236885439050294 is given by 0.749124…. taking 10000000 iterations. Hence Information Dimension is $D_I = 0.749124\ldots$

3. Remarks: We infer that our methods can be extended to higher dimensions for the determination of various fractal dimensions .

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