On the Distribution of Zeros of Polynomials

M. H. Gulzar
Department of Mathematics
University of Kashmir, Srinagar 190006

Abstract: In this paper we find the bounds for the zeros of a polynomial, when the coefficients of the polynomial or their real and imaginary parts are restricted to certain conditions.

Mathematics Subject Classification: 30C10,30C15
Key-words and phrases: Polynomial , Unit disk, Zero.

1.Introduction and Statement of Results

The following elegant result on the distribution of the zeros of polynomials is due to Enestrom and Kakeya [4]:

Theorem A: Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \leq 1$.

In the literature there exist several generalizations and extensions of this result. Recently, M. H. Gulzar [3] proved the following results:

Theorem B: Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, such that for some $k \geq 1$,

either

$$k\alpha_n \geq \alpha_{n-2} \geq ...... \geq \alpha_3 \geq \alpha_1 \geq 0$$

and $\alpha_{n-1} \geq \alpha_{n-3} \geq ...... \geq \alpha_2 \geq \alpha_0 \geq 0$, if n is odd

or

$$k\alpha_n \geq \alpha_{n-2} \geq ...... \geq \alpha_2 \geq \alpha_0 \geq 0$$

$\alpha_{n-1} \geq \alpha_{n-3} \geq ...... \geq \alpha_3 \geq \alpha_1 \geq 0$, if n is even.

Then all the zeros of P(z) lie in the disk

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}\right| \leq \frac{(2k-1)\alpha_n+\alpha_{n-1}+|\beta_n|+|\beta_{n-1}|+2\sum_{j=0}^{n-2}|\beta_j|}{|a_n|} .$$

Theorem C: Let Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j-\beta|\le\alpha\le\beta,\ j=0,1,2,......,n,$$

and for some $k\ge 1$,either

$$k|a_n|\ge|a_{n-2}|\ge......\ge|a_3|\ge|a_1|$$

and $|a_{n-1}|\ge|a_{n-3}|\ge......\ge|a_2|\ge|a_0|$, if n is odd

or

$$k|a_n|\ge|a_{n-2}|\ge......\ge|a_2|\ge|a_0|$$

and $|a_{n-1}|\ge|a_{n-3}|\ge......\ge|a_3|\ge|a_1|$, if n is even.

Then all the zeros of P(z) lie in the disk

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}\right|\le\frac{[k|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+|a_{n-1}|(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|-(|a_1|+|a_0|)(\cos\alpha-\sin\alpha-1)}{|a_n|}.$$

Y. Choo [1] proved the following result:

Theorem D: Let Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j-\beta|\le\alpha\le\beta,\ j=0,1,2,......,n,$$

and for some $k_1\ge 1, k_2\ge 1$,either

$$k_1|a_n|\ge|a_{n-2}|\ge......\ge|a_3|\ge|a_1|$$

and $k_2|a_{n-1}|\ge|a_{n-3}|\ge......\ge|a_2|\ge|a_0|$, if n is odd

or

$$k_1|a_n|\ge|a_{n-2}|\ge......\ge|a_2|\ge|a_0|$$

and $k_2|a_{n-1}|\ge|a_{n-3}|\ge......\ge|a_3|\ge|a_1|$, if n is even.

Then all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K}{|a_n|}$$

where

$$K=(k_1-1)|a_n|+(k_2-1)|a_{n-1}|+|a_1|+|a_0|+(k_1|a_n|+k_2|a_{n-1}|)(\cos\alpha+\sin\alpha)$$
$$+(|a_1|+|a_0|)(\sin\alpha-\cos\alpha)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|.$$

The aim of this paper is to generalize the above results with less restricted conditions on the coefficients. In fact, we prove the following results:

Theorem 1: Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, such that for some $k_1\geq 1, k_2\geq 1$, $0<\tau_1\leq 1, 0<\tau_2\leq 1$,either

$$k_1\alpha_n\geq\alpha_{n-2}\geq......\geq\alpha_3\geq\tau_1\alpha_1$$

and $k_2\alpha_{n-1}\geq\alpha_{n-3}\geq......\geq\alpha_2\geq\tau_2\alpha_0$, if n is odd

or

$$k_1\alpha_n\geq\alpha_{n-2}\geq......\geq\alpha_2\geq\tau_1\alpha_0$$

and $k_2\alpha_{n-1}\geq\alpha_{n-3}\geq......\geq\alpha_3\geq\tau_2\alpha_1$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq\frac{K_1}{|a_n|}$$

where

$$K_1=k_1(\alpha_n+|\alpha_n|)+k_2(\alpha_{n-1}+|\alpha_{n-1}|)+2(|\alpha_1|+|\alpha_0|)-(|\alpha_n|+|\alpha_{n-1}|)-\tau_1(\alpha_1+|\alpha_1|)$$
$$-\tau_2(\alpha_0+|\alpha_0|)+|\beta_n|+|\beta_{n-1}|+2\sum_{j=0}^{n-2}|\beta_j|,$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq\frac{K_2}{|a_n|}$$

where

$$K_2=k_1(\alpha_n+|\alpha_n|)+k_2(\alpha_{n-1}+|\alpha_{n-1}|)+2(|\alpha_1|+|\alpha_0|)-(|\alpha_n|+|\alpha_{n-1}|)-\tau_2(\alpha_1+|\alpha_1|)$$
$$-\tau_1(\alpha_0+|\alpha_0|)+|\beta_n|+|\beta_{n-1}|+2\sum_{j=0}^{n-2}|\beta_j|.$$

Remark 1: Taking $\tau_1=1,\tau_2=1$, $k_1=k, k_2=1$ and the coefficients to be non-negative, we have $K_1=K_2$ and Theorem 1 reduces to Theorem B.

Taking $k_1=1, k_2=1$, the following result is an immediate consequence of Theorem 1:

Corollory 1: Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, such that for some $0<\tau_1\leq 1, 0<\tau_2\leq 1$,either

$$\alpha_n\geq\alpha_{n-2}\geq......\geq\alpha_3\geq\tau_1\alpha_1$$

and $\alpha_{n-1}\geq\alpha_{n-3}\geq......\geq\alpha_2\geq\tau_2\alpha_0$, if n is odd

or

$$\alpha_n\geq\alpha_{n-2}\geq......\geq\alpha_2\geq\tau_1\alpha_0$$

and $\alpha_{n-1}\geq\alpha_{n-3}\geq......\geq\alpha_3\geq\tau_2\alpha_1$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq\frac{K_1}{|a_n|}$$

where

$$K_1 = \alpha_n + \alpha_{n-1} + 2(|\alpha_1| + |\alpha_0|) - \tau_1(\alpha_1 + |\alpha_1|) - \tau_2(\alpha_0 + |\alpha_0|) + |\beta_n| + |\beta_{n-1}| + 2\sum_{j=0}^{n-2}|\beta_j|,$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{K_2}{|a_n|}$$

where

$$K_2 = \alpha_n + \alpha_{n-1} + 2(|\alpha_1| + |\alpha_0|)) - \tau_2(\alpha_1 + |\alpha_1|) - \tau_1(\alpha_0 + |\alpha_0|) + |\beta_n| + |\beta_{n-1}| + 2\sum_{j=0}^{n-2}|\beta_j|.$$

If the coefficients a_j are real i.e. $\beta_j = 0$ for all j, Theorem 1 gives the following result:

Corollory 2: Let Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $0 < \tau_1 \le 1, 0 < \tau_2 \le 1$,either

$$a_n \ge a_{n-2} \ge ...... \ge a_3 \ge \tau_1 a_1$$

and $a_{n-1} \ge a_{n-3} \ge ...... \ge a_2 \ge \tau_2 a_0$, if n is odd

or

$$a_n \ge a_{n-2} \ge ...... \ge a_2 \ge \tau_1 a_0$$

and $a_{n-1} \ge a_{n-3} \ge ...... \ge a_3 \ge \tau_2 a_1$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{K_1}{|a_n|}$$

where

$$K_1 = a_n + a_{n-1} + 2(|a_1| + |a_0|) - \tau_1(a_1 + |a_1|) - \tau_2(a_0 + |a_0|),$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{K_2}{|a_n|}$$

where

$$K_2 = a_n + a_{n-1} + 2(|a_1| + |a_0|)) - \tau_2(a_1 + |a_1|) - \tau_1(a_0 + |a_0|).$$

Applying Theorem 1 to the polynomial –iP(z), we get the following result:

Corollory 3: Let Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j = \alpha_j + i\beta_j, j = 0,1,2,......,n$, such that for some $0 < \tau_1 \le 1, 0 < \tau_2 \le 1$,either

$$k_1\beta_n \ge \beta_{n-2} \ge ...... \ge \beta_3 \ge \tau_1\beta_1$$

and $k_2\beta_{n-1} \ge \beta_{n-3} \ge ...... \ge \beta_2 \ge \tau_2\beta_0$, if n is odd

or

$$k_1\beta_n \ge \alpha\beta_{n-2} \ge ...... \ge \beta_2 \ge \tau_1\beta_0$$

and $k_2\beta_{n-1} \ge \beta_{n-3} \ge ...... \ge \beta_3 \ge \tau_2\beta_1$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_1}{|a_n|}$$

where

$$K_1 = k_1(\beta_n + |\beta_n|) + k_2(\beta_{n-1} + |\beta_{n-1}|) + 2(|\beta_1| + |\beta_0|) - \tau_1(\beta_1 + |\beta_1|)$$
$$- \tau_2(\beta_0 + |\beta_0|) + |\alpha_n| + |\alpha_{n-1}| + 2\sum_{j=0}^{n-2}|\alpha_j|$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_2}{|a_n|}$$

where

$$K_2 = k_1(\beta_n + |\beta_n|) + k_2(\beta_{n-1} + |\beta_{n-1}|) + 2(|\beta_1| + |\beta_0|) - \tau_2(\beta_1 + |\beta_1|)$$
$$- \tau_1(\beta_0 + |\beta_0|) + |\alpha_n| + |\alpha_{n-1}| + 2\sum_{j=0}^{n-2}|\alpha_j|$$

Theorem 2: Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j - \beta| \le \alpha \le \beta,\ j = 0,1,2,......,n,$$

and for some $k_1 \ge 1$, $k_2 \ge 1, 0 < \tau_1 \le 1, 0 < \tau_2 \le 1$, either

$$k_1|a_n| \ge |a_{n-2}| \ge ...... \ge |a_3| \ge \tau_1|a_1|$$

and $k_2|a_{n-1}| \ge |a_{n-3}| \ge ...... \ge |a_2| \ge \tau_2|a_0|$, if n is odd

or

$$k_1|a_n| \ge |a_{n-2}| \ge ...... \ge |a_2| \ge \tau_1|a_0|$$

and $k_2|a_{n-1}| \ge |a_{n-3}| \ge ...... \ge |a_3| \ge \tau_2|a_1|$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_1}{|a_n|}$$

where

$$K_1 = (k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) - (|a_n| + |a_{n-1}|)$$
$$- (\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2(|a_1| + |a_0|) + 2\sin\alpha\sum_{j=2}^{n-2}|a_j|.$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_2}{|a_n|}$$

where

$$K_2 = (k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) - (|a_n| + |a_{n-1}|)$$

$$-(\tau_1|a_0|+\tau_2|a_1|)(\cos\alpha-\sin\alpha+1)+2(|a_1|+|a_0|)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|.$$

Remark 2: Taking $\tau_1 = 1, \tau_2 = 1$, Theorem 2 reduces to Theorem D.
Taking $k_1 = 1, k_2 = 1$, Theorem 2 gives the following result:

Corollory 4: Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j - \beta| \le \alpha \le \beta,\ j = 0,1,2,......,n,$$

and for some $k_1 \ge 1$, $k_2 \ge 1, 0<\tau_1 \le 1, 0<\tau_2 \le 1$, either

$$|a_n| \ge |a_{n-2}| \ge ...... \ge |a_3| \ge \tau_1|a_1|$$

and $|a_{n-1}| \ge |a_{n-3}| \ge ...... \ge |a_2| \ge \tau_2|a_0|$, if n is odd

or

$$|a_n| \ge |a_{n-2}| \ge ...... \ge |a_2| \ge \tau_1|a_0|$$

and $|a_{n-1}| \ge |a_{n-3}| \ge ...... \ge |a_3| \ge \tau_2|a_1|$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_1}{|a_n|}$$

where

$$K_1 = (|a_n|+|a_{n-1}|)(\cos\alpha+\sin\alpha)-(\tau_1|a_1|+\tau_2|a_0|)(\cos\alpha-\sin\alpha+1)+2(|a_1|+|a_0|)$$
$$+2\sin\alpha\sum_{j=2}^{n-2}|a_j|$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_2}{|a_n|}$$

where

$$K_2 = (|a_n|+|a_{n-1}|)(\cos\alpha+\sin\alpha)-(\tau_1|a_0|+\tau_2|a_1|)(\cos\alpha-\sin\alpha+1)+2(|a_1|+|a_0|)$$
$$+2\sin\alpha\sum_{j=2}^{n-2}|a_j|$$

2.Lemmas

For the proofs of the above theorems, we need the following result:

Lemma 1: . Let P(z)=$\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, j = 0,1,...n,$ *for* some real β and

$$|a_j| \ge |a_{j-1}|,\ j = 1,2,.......,n,$$

then for some t>0,

$$|ta_j - a_{j-1}| \le \left[t|a_j| - |a_{j-1}|\right]\cos\alpha + \left[t|a_j| + |a_{j-1}|\right]\sin\alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [2].

3. Proofs of Theorems

Proof of Theorem 1: Let n be odd. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z^2)P(z)\\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1}-a_{n-3})z^{n-1} + (a_{n-2}-a_{n-4})z^{n-2}\\
&\quad + ...+(a_4 - a_2)z^4 + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0\\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (1-k_1)a_n z^n + (1-k_2)a_{n-1}z^{n-1} + a_1 z + a_0\\
&\quad + (k_1 a_n - a_{n-2})z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2}\\
&\quad + ...+(a_4 - a_2)z^4 + (a_3 - \tau_1 a_1)z^3 + (\tau_1 a_1 - a_1)z^3 + (a_2 - \tau_2 a_0)z^2\\
&\quad + (\tau_2 a_0 - a_0)z^2.\\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (1-k_1)\alpha_n z^n + (1-k_2)\alpha_{n-1}z^{n-1} + \alpha_1 z + \alpha_0\\
&\quad + (k_1\alpha_n - \alpha_{n-2})z^n + (k_2\alpha_{n-1} - \alpha_{n-3})z^{n-1} + (\alpha_{n-2} - \alpha_{n-4})z^{n-2}\\
&\quad + ...+(\alpha_4 - \alpha_2)z^4 + (\alpha_3 - \tau_1\alpha_1)z^3 + (\tau_1\alpha_1 - \alpha_1)z^3 + (\alpha_2 - \tau_2\alpha_0)z^2\\
&\quad + (\tau_2\alpha_0 - \alpha_0)z^2 + i\{(\beta_n - \beta_{n-2})z^n + (\beta_{n-1} - \beta_{n-3})z^{n-1} + ...+(\beta_2 - \beta_0)z^2\\
&\quad + \beta_1 z + \beta_0\}
\end{aligned}$$

For $|z| > 1$,

$$\begin{aligned}
|F(z)| &\ge |z|^{n+1}\Bigg[|a_n z + a_{n-1}| - \{(k_1 - 1)|\alpha_n|\frac{1}{|z|} + (k_2 - 1)|\alpha_{n-1}|\frac{1}{|z|^2} + \frac{|\alpha_1|}{|z|^n} + \frac{|\alpha_0|}{|z|^{n+1}} + |k_1\alpha_n - \alpha_{n-2}|\frac{1}{|z|}\\
&\quad + |k_2\alpha_{n-1} - \alpha_{n-3}|\frac{1}{|z|^2} + |\alpha_{n-2} - \alpha_{n-4}|\frac{1}{|z|^3} + |\alpha_{n-3} - \alpha_{n-5}|\frac{1}{|z|^4} + ...\\
&\quad + |\alpha_4 - \alpha_2|\frac{1}{|z|^{n-3}} + |\alpha_3 - \tau_1\alpha_1|\frac{1}{|z|^{n-2}} + (1-\tau_1)|\alpha_1|\frac{1}{|z|^{n-2}} + |\alpha_2 - \tau_2\alpha_0|\frac{1}{|z|^{n-1}}\\
&\quad + (1-\tau_2)|\alpha_0|\frac{1}{|z|^{n-1}} + |\beta_n - \beta_{n-2}|\frac{1}{|z|} + ...+|\beta_2 - \beta_0|\frac{1}{|z|^{n-1}} + \frac{|\beta_1|}{|z|^n} + \frac{|\beta_0|}{|z|^{n+1}}\ \}\Bigg]\\
&> |z|^{n+1}\Big[|a_n z + a_{n-1}| - \{(k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + |\alpha_1| + |\alpha_0| + k_1\alpha_n - \alpha_{n-2}\\
&\quad + k_2\alpha_{n-1} - \alpha_{n-3} + \alpha_{n-2} - \alpha_{n-4} + ...+\alpha_4 - \alpha_2 + \alpha_3 - \tau_1\alpha_1\\
&\quad + (1-\tau_1)|\alpha_1| + \alpha_2 - \tau_2\alpha_0 + (1-\tau_2)|\alpha_0| + |\beta_n| + |\beta_{n-2}| + ...+|\beta_1| + |\beta_0|\}\Big]\\
&= |z|^{n+1}\Big[|a_n z + a_{n-1}| - \{(k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + k_1\alpha_n + k_2\alpha_{n-1}\\
&\quad - \tau_1(\alpha_1 + |\alpha_1|) - \tau_2(\alpha_0 + |\alpha_0|) + 2(|\alpha_0| + |\alpha_1|) + 2(|\beta_0| + |\beta_1|) + \sum_{j=2}^{n}|\beta_j|\ \}\Big]\\
&= |z|^{n+1}\Big[|\alpha_n z + \alpha_{n-1}| - \{k_1(\alpha_n + |\alpha_n|) + k_2(\alpha_{n-1} + |\alpha_{n-1}|) - (|\alpha_n| + |\alpha_{n-1}|)\\
&\quad - \tau_1(\alpha_1 + |\alpha_1|) - \tau_2(\alpha_0 + |\alpha_0|) + 2(|\alpha_0| + |\alpha_1|) + 2(|\beta_0| + |\beta_1|) + \sum_{j=2}^{n}|\beta_j|\ \}\Big]
\end{aligned}$$

$$> 0$$

if

$$\left| z + \frac{a_{n-1}}{a_n} \right| > \frac{K_1}{|a_n|},$$

where

$$K_1 = .k_1(\alpha_n + |\alpha_n|) + k_2(\alpha_{n-1} + |\alpha_{n-1}|) - (|\alpha_n| + |\alpha_{n-1}| - \tau_1(\alpha_1 + |\alpha_1|) - \tau_2(\alpha_0 + |\alpha_0|) + 2(|\alpha_0| + |\alpha_1|)$$
$$+ 2(|\beta_0| + |\beta_1|) + \sum_{j=2}^{n} |\beta_j|$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \le \frac{K_1}{|a_n|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence it follows that all the zeros of F(z) and therefore P(z) lie in

$$\left| z + \frac{a_{n-1}}{a_n} \right| \le \frac{K_1}{|a_n|}.$$

A similar argument applies to the case when n is even and Theorem 1 follows.

Proof of Theorem 2. Let n be odd. Consider the polynomial

$$\begin{aligned}
F(z) &= (1 - z^2)P(z) \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2} \\
&\quad + ...+ (a_4 - a_2)z^4 + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0 \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (1 - k_1)a_n z^n + (1 - k_2)a_{n-1}z^{n-1} + a_1 z + a_0 \\
&\quad + (k_1 a_n - a_{n-2})z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2} \\
&\quad + ...+ (a_4 - a_2)z^4 + (a_3 - \tau_1 a_1)z^3 + (\tau_1 a_1 - a_1)z^3 + (a_2 - \tau_2 a_0)z^2 \\
&\quad + (\tau_2 a_0 - a_0)z^2.
\end{aligned}$$

For $|z| > 1$, we have , by using the lemma,

$$\begin{aligned}
|F(z)| &\ge |z|^n \Big[|a_n z + a_{n-1}| - \{(k_1 - 1)|a_n| + (k_2 - 1)|a_{n-1}| + |a_1| + |a_0| + |k_1 a_n - a_{n-2}| \\
&\quad + |k_2 a_{n-1} - a_{n-3}| + |a_{n-2} - a_{n-4}| + |a_{n-3} - a_{n-5}| + ... \\
&\quad + |a_4 - a_2| + |a_3 - \tau_1 a_1| + |(1 - \tau_1)a_1| + |a_2 - \tau_2 a_0| + |(1 - \tau_2)a_0| \ \} \ \Big] \\
&\ge |z|^n \Big[|a_n z + a_{n-1}| - \{(k_1 - 1)|a_n| + (k_2 - 1)|a_{n-1}| + |a_1| + |a_0| + (k_1|a_n| - |a_{n-2}|)\cos\alpha \\
&\quad + (k_1|a_n| + |a_{n-1}|)\sin\alpha + (k_2|a_{n-1}| - |a_{n-3}|)\cos\alpha + (k_2|a_{n-1}| + |a_{n-3}|)\sin\alpha \\
&\quad + (|a_{n-2}| - |a_{n-4}|)\cos\alpha + (|a_{n-2}| + |a_{n-4}|)\sin\alpha + ...+ (|a_4| - |a_2|)\cos\alpha \\
&\quad + (|a_4| + |a_2|)\sin\alpha + (|a_3| - \tau_1|a_1|)\cos\alpha + (|a_3| + \tau_1|a_1|)\sin\alpha \\
&\quad + (|a_2| - \tau_2|a_0|)\cos\alpha + (|a_2| + \tau_2|a_0|)\sin\alpha + (1 - \tau_1)|a_1| + (1 - \tau_2)|a_0| \\
&= |z|^n \Big[|a_n z + a_{n-1}| - \{(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) - (|a_n| + |a_{n-1}|)
\end{aligned}$$

$$-(\tau_1|a_1|+\tau_2|a_0|)(\cos\alpha-\sin\alpha+1)+2(|a_1|+|a_0|)+2\sin\alpha\sum_{j=2}^{n-2}|a_j| \ \} \ ]$$

$$>0$$

if

$$\left|z+\frac{a_{n-1}}{a_n}\right|>\frac{K_1}{|a_n|},$$

where

$$K_1=(k_1|a_n|+k_2|a_{n-1}|)(\cos\alpha+\sin\alpha+1)-(|a_n|+|a_{n-1}|)$$
$$-(\tau_1|a_1|+\tau_2|a_0|)(\cos\alpha-\sin\alpha+1)+2(|a_1|+|a_0|)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|.$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K_1}{|a_n|}.$$

But the zeros of F(z) having modulus less than or equal to 1 already satisfy the above inequality. Hence it follows that all the zeros of F(z) and therefore P(z) lie in

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K_1}{|a_n|}.$$

The case when n is even can be proved similarly.

References

[1] Y. Choo, On the Distribution of Zeros of Polynomial, Int. Journal of Math. Analysis, Vol.6, 2012,No.42, 2091-2099.
[2] N. K. Govil and Q. I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math. J.20(1968) , 126-136.
[3] M.H.Gulzar, On the Location of Zeros of Polynomials with Restricted Coefficients, International Journal of Advanced Scientific Research and Technology,Issue 2, Vol. 3, June 2012,546-556.
[4] M.Marden, Geometry of Polynomials, Math. Surveys No.3,Amer. Math. Soc.(R.I.Providence),1966.