On The Numerical Solution of Volterra-Fredholm Integral Equations With Abel Kernel Using Legendre Polynomials

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Abstract

Legendre collocation method and Trapezoidal rule are presented to solve numerically the Voltterra-Fredholm Integral Equations with Abel kernel. We transform the Volterra integral equations to a system of Fredholm integral equations of the second kind which will be solved by Legendre method. This method is based on replacement of the unknown function by truncated series. This lead to a system of algebraic equations. Thus, by solving the matrix equation, the coefficients are obtained. A numerical example is included to certify the validity and applicability of the proposed technique.

Keywords: Volterra-Fredholm Integral Equations, Abel kernel, Integral equation, collocation matrix method, Legendre polynomials, Trapezoidal rule.

1 Introduction

We consider the Volterra-Fredholm integral equation of the second kind with Abel kernel:

$$\psi(x,t) - \int_0^t \int_{-1}^1 |x-y|^{-\alpha} \psi(y,\tau) dy d\tau = f(x,t),\ (x,t) \in [-1,1] \times [0,T] \qquad (1)$$

where $0 \leqslant T$ is given and $\alpha \in]0,1[$. The elements $\mathcal{K}(x,y) = |x-y|^{-\alpha}$ is the Abel kernel.

For solving Volterra-Fredholm integral equations, many methods with enough accuracy and efficiency have been used before by many researches [1, 1, 3, 4, 5, 6, 7, 8, 9]. Maleknejad and Fadaei Yami [5] solved the system of Volterra-Fredholm integral equations by Adomian decomposition method. Kauthen in [4], used continuous time collocation method for Volterra-Fredholm integral equations. Legendre

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wavelets also were applied for solving Volterra-Fredholm integral equations [10]. In [11], Yalsinbas developed numerical solution of nonlinear Volterra-Fredholm integral equations by using Taylor polynomials. In this paper, we use numerical technique based on Trapezoidal rule, to reduce the Volterra-integral Equations to a linear system of Fredhom Integal equations which will be solved using Legendre collocation method. The paper is organized as follows. In section 2, we transform the Volterra-fredholm Integral equations to a system of Fredholm integral equations of the second kind. In Section 3, a rapid review of Legendre polyomial and a linear system is obtained, so an approximation solution is presented with convergence theorem. In the remainder of the paper, we give a practical example to certify the validity of the proposed technique.

2 System of Fredholm Integral Equations

First, if $t = 0$ the Volterra-Fredholm integral equations is reduced to: $\psi(x,0) = f(x,0)$. For $t \neq 0$, we apply Trapezoidal Method to solve the Volterra integral equations according to the variable τ. For a given t, we divide the interval of integration $(0;t)$ into m equal subintervals, $\delta\tau = \frac{t_m - 0}{m}$, where $t_m = t$.

Let $\tau_0 = 0, t_0 = \tau_0, t_m = \tau_m = t, \tau_j = j\delta\tau, t_j = \tau_j$. Using the trapezoid rule,

$$\int_0^t \int_{-1}^1 |x-y|^{-\alpha}\psi(y,\tau)dyd\tau \sim \delta\tau \sum_{j=0}^{m}{}'' \int_{-1}^1 |x-y|^{-\alpha}\psi(y,\tau_j)d\,y$$

where the double prime indicates that the first and last term to be halved, where

$$\delta\tau = \frac{\tau_j - 0}{j} = \frac{t-0}{m}, \tau_j \leqslant t, j \geqslant 1, t = t_m = \tau_m$$

In all our approximation, the error assumed negligible, this help us to get a system of Fredholm Integral equations.

Now, for $0 \leqslant r \leqslant m$, the Volterra Fredholm integral equations become a system of Fredholm integral equations

$$\psi(x,t_r) - \delta\tau \sum_{j=0}^{r}{}'' \int_{-1}^1 |x-y|^{-\alpha}\psi(y,\tau_j)d\,y = f(x,t_r),\ 1 \leqslant r$$

and $\psi(x,0) = f(x,0)$.

We get the system:

$$\begin{aligned}
\psi(x,0) &= f(x,0)\\
\psi(x,t_1) - \frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,t_1)dy &= f(x,t_1) + \frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,0)dy\\
\psi(x,t_2) - \frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,t_2)dy &= f(x,t_2) + \frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,0)dy + \delta\tau\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,t_1)dy\\
&\vdots\\
\psi(x,t_m) - \frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,t_m)dy &= f(x,t_m) + \delta\tau\sum_{j=0}^{m-1\,'}\int_{-1}^{1}|x-y|^{-\alpha}\psi(y,\tau_j)d\,y,
\end{aligned}$$

where the prime indicates that the first term to be halved. Denote: $f(x,t_\ell) = f^\ell(x), \psi(y,\tau_\ell) = \psi^\ell(y), \ell = 0,\ldots,m$

Putting

$$F^m(x) = f^m(x) + \sum_{j=0}^{m-1\,'}\int_{-1}^{1}|x-y|^{-\alpha}\psi^j(y)d\,y,$$

An obvious computation gives

$$F^m(x) = f^m(x) + 2\sum_{j=1}^{m-1}(-1)^{j+m}\Big(f^j(x) - \psi^j(x)\Big) + (-1)^{m+1}\frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi^0(y)dy$$

Now, our problem become:

$$\begin{aligned}
\psi^\ell(x) - \frac{\delta\tau}{2}\int_{-1}^{1}|x-y|^{-\alpha}\psi^\ell(y)dy &= F^\ell(x),\ \ell = 1,\ldots,m \qquad (2)\\
\psi(x,0) &= f(x,0)
\end{aligned}$$

Equations (2) represents a system of Fredholm integral equations of the second kind which will be solved by Legendre method. For a fixed t_ℓ, we solve the Fredholm integral equations which leads to the required approximate solution of the volterra-Fredholm integral equation (1).

3 Legendre Methods

3.1 Fundamental

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common set of orthogonal polynomials is the Legendre polynomials. The Legendre polynomials P_n satisfy the recurrence formula:

$$\begin{aligned} (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x),\ n \in \mathbb{N}^* \\ P_0(x) &= 1, \\ P_1(x) &= x \end{aligned} \tag{3}$$

An important property of the Legendre polynomials is that they are orthogonal with respect to the L^2 inner product on the interval $[-1, 1]$:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$

where δ_{nm} denotes the Kronecker delta.

3.2 Approximate solution

We choose $x_k, k \in \{0, \ldots, n\}$ the zeros of the Legendre polynomial of degree equal $n+1$. We determine a suitable interpolating elements $\phi_j^{\ell}(x), j = 0, 1, \ldots, n$, such that

$$\psi_n^{\ell}(x) = \sum_{j=0}^{n} \phi_j^{\ell}(x)\psi^{\ell}(x_j) \tag{4}$$

is the unique interpolating polynomial of degree n, which interpolates ψ^{ℓ} at the points $x_i, i = 0, 1, \ldots, n$.

The elements $\phi_j(x), j = 0, 1, \ldots, n$ are called the basic functions associated with the Legendre interpolation polynomial and they satisfy $\phi_j(x_i) = \delta_{ij}$.

Then we get an approximation of the exactly integral, let say:

$$I_n(\psi^{\ell}) = \int_{-1}^{1} \mathcal{K}(x,y)\psi_n^{\ell}(y)dy \tag{5}$$

This type of approximation must be chosen so that the integral (5) can be evaluated (either explicitly or by an efficient numerical technique).

The functions $P_0(x), P_1(x), \ldots, P_n(x)$ will be called interpolating elements. In this dissertation, the interpolating function ψ_n^{ℓ} will be assumed to be the interpolating polynomial

$$\psi_n^{\ell}(x) = \sum_{j=0}^{n} \beta_j^{\ell} P_j(x) \tag{6}$$

where P_j are Legendre polynomials of degree j, n is the number of Legendre polynomials, and β_j^{ℓ} are unknown parameters, to be determined. This

The coefficients β_j^{ℓ} are obtained by multiplying both sides of Eq. (6) by $P_m, m \leq n$ (as weight functions), and integrating the resulting equation with respect to x over the interval $[-1, 1]$ to obtain

$$\int_{-1}^{1} P_m(x)\psi_n^\ell(x)dx = \sum_{j=0}^{n}\beta_j^\ell \int_{-1}^{1} P_m(x)P_j(x)dx = \beta_m^\ell \frac{2}{2m+1}$$

Therefore,

$$\beta_m^\ell = \frac{2m+1}{2}\int_{-1}^{1} P_m(x)\psi_n^\ell(x)dx \tag{7}$$

Here the integrand $P_m\psi_n^\ell$ is a polynomial of degree $n+m \leq 2n$ then its integration in (7) can exactly be obtained from just $n+1$ point Gauss-Legendre method, by using the following formula

$$\beta_m^\ell = \frac{2m+1}{2}\sum_{j=0}^{n} w_j P_m(x_j)\psi^\ell(x_j) \tag{8}$$

where $w_j, j = 0, \ldots, n$ are the $(n+1)$-point Gauss-Legendre weights(The w_j are given coefficients that do not depend on the integrant function).

The $n+1$ grid points (x_i) of Gauss Legendre integration in formula (8) giving us the exact integral of an integrand polynomial of degree $n+m \leq 2n$ can be obtained as the zeros of the $n+1$-th-degree Legendre polynomial. Then, given the $n+1$ grid point x_i, we can get the corresponding weight w_i of the i point Gauss Legendre integration formula by solving the system of linear equations. Now, the interpolating polynomial ψ_n^ℓ can be written as:

$$\begin{aligned}\psi_n^\ell(x) &= \sum_{m=0}^{n}\Big(\frac{2m+1}{2}\sum_{j=0}^{n} w_j P_m(x_j)\psi^\ell(x_j)\Big)P_m(x)\\ &= \sum_{j=0}^{n}\Big(w_j\sum_{m=0}^{n}\frac{2m+1}{2}P_m(x_j)P_m(x)\Big)\psi^\ell(x_j)\end{aligned} \tag{9}$$

Using (4) and (9) we get

$$\phi_j^\ell(x) = w_j\sum_{m=0}^{n}\frac{2m+1}{2}P_m(x_j)P_m(x),\ j = 0, \ldots, n \tag{10}$$

Substituting ψ_n^ℓ into Eq. (2) and collocating at the points x_i, we obtain:

$$\psi^\ell(x_i) - \frac{\delta\tau}{2}\sum_{j=0}^{n}\psi^\ell(x_j)\int_{-1}^{1}\mathcal{K}(x_i, y)\phi_j^\ell(y)dy = F^\ell(x_i), i = 0, \ldots, n \tag{11}$$

3.3 Convergence

We define

$$H^m([-1,1]) = \Big\{\phi \in L^2[-1,1], 0 \leq k \leq m, \frac{d^k\phi}{dx^k} \in L^2[-1,1]_Big\}$$

the spaces $H^m([-1,1])$ endowed with the following inner product are Hilbert spaces

$$< \phi, \psi >_m = \sum_{k=0}^{m} \int_{-1}^{1} \frac{d^k\phi}{dx^k}(x)\frac{d^k\psi}{dx^k}(x)dx$$

with the associated norm

$$||\phi||_m = \left(\sum_{k=0}^{m} ||\frac{d^k\phi}{dx^k}||^2_{L^2[-1,1]} \right)^{1/2}$$

Theorem 1 *Let $\psi^\ell \in H^m[-1,1]$, then ψ^ℓ_n (The truncated Legendre series) is the best approximation polynomial of ψ^ℓ. Moreever, $\exists C > 0$, such that*

$$||\psi^\ell - \psi^\ell_n||_{L^2[-1,1]} \leqslant Cn^{-m}||\psi^\ell||_{H^m[-1,1]}$$

Proof 1 *See [13, 12]*

3.4 Matrix Form

To simplify the presentation let us define

$$a_{i,j} = \int_{-1}^{1} \mathcal{K}(x_i, y)\phi^\ell_j(y)dy \tag{12}$$

Then a $(n+1) \times (n+1)$ linear system is obtained:

$$(Id - \frac{\delta\tau}{2}A^\ell)\psi^\ell = F^\ell \tag{13}$$

where $A^\ell = (a_{i,j})_{(i,j)\in\{0,\ldots,n\}^2}$ is square matrix, $\psi^\ell = (\psi^\ell(x_0), \ldots, \psi^\ell(x_n))^T$ and $F^\ell = (F^\ell(x_0), \ldots, F^\ell(x_n))^T$, capital T indicate the transpose. Obviously, the system (13) has a unique solution if the determinant of the matrix $Id - \frac{\delta\tau}{2}A$ is nonzero, which also depends on the choice of collocation point.

Substituting (10) into (12) we obtain

$$a_{i,j} = w_j \sum_{k=0}^{n} \frac{2k+1}{2} P_k(x_j)u_k(x_i)$$

where $u_k(x_i), (i,k) \in \{0,\ldots,n\}^2$ are defined

$$u_k(x_i) = \int_{-1}^{1} |x_i - y|^{-\alpha} P_k(y)dy$$

The constants $u_k(x_i), (i,k) \in \{0,\ldots,n\}^2$, can be evaluated from the recurrence relation:

$$(k+3-\alpha)u_{k+2}(x_i) = (2k+3)x_i u_{k+1}(x_i) - (k+\alpha)u_k(x_i), k = 0,\ldots,n$$

with the starting values for this recurrence relations are:

$$u_0(x_i) = \frac{1}{1-\alpha}\Big((1-x_i)^{1-\alpha} + (1+x_i)^{1-\alpha}\Big) \tag{14}$$

$$u_1(x_i) = x_i u_0(x_i) + \frac{1}{2-\alpha}\Big((1-x_i)^{2-\alpha} + (1+x_i)^{2-\alpha}\Big) \tag{15}$$

4 Numerical Example

In this section, to achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results. The computations, associated with the example, are performed by MATLAB 7. In our computation we will take $\alpha = \frac{1}{2}$, $n = 9$, $m = 9$ *and* $T = 2$. *Consider the Volterra-Fredholm Integral equation of second kind with:*

$f(x,t) = \frac{x^3}{1+t} - \ln(1+t)\Big(w_1 + w_2\Big)$

where

$$\begin{aligned} w_1 &= -\frac{1}{7}x_1^7 + \frac{3}{5}xx_1^5 - x^2x_1^3 + x^3x_1 \\ w_2 &= \frac{1}{7}x_2^7 + \frac{3}{5}xx_2^5 + x^2x_2^3 + x^3x_2 \\ x_1 &= \sqrt{1+x} \\ x_2 &= \sqrt{1-x} \end{aligned}$$

and the exact solution is $\psi(x,t) = \frac{x^3}{1+t}$.

The numerical and exact solutions are compared by considering the absolute error $|\psi(x,t) - \psi_n(x,t)|$. *The Table 1 show that the proposed approach can be a suitable method for solving Volterra-Fredholm integral equations numerically.*

x_k \ t_k	0	0.2222	0.4444	0.6667	0.8889	1.1111	1.3333	1.5556	1.7778	2.0000
-0.9739	0	0.0510	0.1609	0.2945	0.4652	0.6906	0.9956	1.4149	1.9983	2.8173
-0.8651	0	0.0502	0.1616	0.3033	0.4918	0.7491	1.1065	1.6084	2.3187	3.3294
-0.6794	0	0.0347	0.1147	0.2237	0.3772	0.5974	0.9160	1.3790	2.0536	3.0376
-0.4334	0	0.0164	0.0564	0.1156	0.2050	0.3403	0.5450	0.8536	1.3173	2.0117
-0.1489	0	0.0041	0.0149	0.0321	0.0597	0.1031	0.1711	0.2762	0.4376	0.6835
0.1489	0	0.0041	0.0149	0.0321	0.0597	0.1031	0.1711	0.2762	0.4376	0.6835
0.4334	0	0.0164	0.0564	0.1156	0.2050	0.3403	0.5450	0.8536	1.3173	2.0117
0.6794	0	0.0347	0.1147	0.2237	0.3772	0.5974	0.9160	1.3790	2.0536	3.0376
0.8651	0	0.0502	0.1616	0.3033	0.4918	0.7491	1.1065	1.6084	2.3187	3.3294
0.9739	0	0.0510	0.1609	0.2945	0.4652	0.6906	0.9956	1.4149	1.9983	2.8173

Table 1: $10^2|\psi(x,t) - \psi_9(x,t)|$.

5 Conclusion

We used the Trapezoidal rule and Legendre expansion to approximate the numerical solution of Volterra-Fredholm integral equations, this allows us to reduce the Volterra- Fredholm integral equation to a system of linear equations. According to the numerical results which obtaining from the illustrative example, we conclude that we have a good accuracy. This method may be applied to solve Volterra Fredholm integral equations with other singular Kernels (logarithm Kernel), smooth kernels and a nonlinear Volterra Fredholm integral equation.

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