If f is entire then clearly,

$$\rho_f^{*L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]}[r\exp\{L(r)\}]}.$$

The following definition is also well known.

Definition 4 *The hyper L^*-order $\overline{\rho}_f^{L^*}$ of a meromorphic function f is defined as follows:*

$$\overline{\rho}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log[r\exp\{L(r)\}]}.$$

If f is entire then

$$\overline{\rho}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log[r\exp\{L(r)\}]}.$$

In this paper we introduce the following definitions.

Definition A *The L^*-type $\sigma_f^{*L^*}$ of a meromorphic function of L^*-order zero is defined by*

$$\sigma_f^{*L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{[\log[r\exp\{L(r)\}]]^{\rho_f^{*L^*}}}, \quad 0 < \rho_f^{*L^*} < \infty.$$

Definition B *A meromorphic function f of L^*-order zero is said to be of L^*-type $\sigma_f^{*L^*}$ if the integral*

$$\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} \quad (r_0 > 0)$$

is convergent for $k > \sigma_f^{*L^*}$ and divergent for $k < \sigma_f^{*L^*}$ where $0 < \rho_f^{*L^*} < \infty$.

Definition C *The L^*-type $\overline{\sigma}_f^{L^*}$ of a meromorphic function of L^*-order infinity is defined as follows:*

$$\overline{\sigma}_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{[r\exp\{L(r)\}]^{\overline{\rho}_f^{L^*}}}, \text{where } 0 < \overline{\rho}_f^{L^*} < \infty.$$

Definition D *A meromorphic function f of L^*-order infinity is said to be of L^*-type $\overline{\sigma}_f^{L^*}$ if the integral*

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} \quad (r_0 > 0)$$

converges for $k > \overline{\sigma}_f^{L^*}$ and diverges for $k < \overline{\sigma}_f^{L^*}$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *Let the integral*

$$\int_{r_0}^{\infty} \frac{\exp[T(r,f)]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} \quad (r_0 > 0) \tag{A}$$

converges for $0 < k < \infty$. *Then*

$$\lim_{r\to\infty} \frac{\exp[T(r,f)]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k}} = 0.$$

Proof. Since the integral

$$\int_{r_0}^{\infty} \frac{\exp[T(r,f)]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}}$$

is convergent for $0 < k < \infty$, given $\epsilon\,(> 0)$ there exists a number $R = R(\epsilon)$ such that

$$\int_{r_0}^{\infty} \frac{\exp[T(r,f)]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} < \epsilon \text{ for } r_0 > R$$

i.e., for $r_0 > R$,

$$\int_{r_0}^{r_0+\exp[\log\{r_0e^{L(r_0)}\}]^{\rho_f^{*L^*}}} \frac{\exp[T(r,f)]dr}{\left[\exp\left\{\log\left(r_0e^{L(r_0)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} < \epsilon.$$

As $\exp[T(r,f)]$ is an increasing function of r, so

$$\begin{aligned}
&\int_{r_0}^{r_0+\exp[\log\{r_0e^{L(r_0)}\}]^{\rho_f^{*L^*}}} \frac{\exp[T(r,f)]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} \\
\geq\ & \frac{\exp[T(r_0,f)]}{\left[\exp\left\{\log\left(r_0e^{L(r_0)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} . \exp[\log\{r_0e^{L(r_0)}\}]^{\rho_f^{*L^*}} \\
=\ & \frac{\exp[T(r_0,f)]}{\left[\exp\left\{\log\left(r_0e^{L(r_0)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k}}
\end{aligned}$$

i.e.,

$$\frac{\exp[T(r_0,f)]}{\left[\exp\left\{\log\left(r_0e^{L(r_0)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k}} < \epsilon \quad \text{for} \quad r_0 > R,$$

from which it follows that

$$\limsup_{r\to\infty} \frac{\exp[T(r,f)]}{\left[\exp\left\{\log\left(r_0 e^{L(r_0)}\right)\right\}^{\rho_f^{*L^*}}\right]^k} = 0.$$

This proves the lemma. ■

Lemma 2 *If the integral*

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k+1}} \quad (r_0 > 0)$$

is convergent for $0 < k < \infty$ *then*

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^k} = 0.$$

Proof. Since the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k+1}}$$

converges for $0 < k < \infty$, given $\epsilon\,(> 0)$ there exists a number $R = R(\epsilon)$ such that

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k+1}} < \epsilon \text{ for } r > R.$$

i.e.,

$$\begin{aligned} &\int_{r_0}^{r_0+\exp\left[r_0 e^{L(r_0)}\right]^{\overline{\rho}_f^{L^*}}} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k+1}} \\ \geq\ & \frac{T(r_0,f).\exp\left[r_0 e^{L(r_0)}\right]^{\overline{\rho}_f^{L^*}}}{\left[\exp\left\{r_0 e^{L(r_0)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k+1}} \\ =\ & \frac{T(r_0,f)}{\left[\exp\left\{r_0 e^{L(r_0)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k}} \end{aligned}$$

i.e., $$\frac{T(r_0,f)}{\left[\exp\left\{r_0 e^{L(r_0)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{k}} < \epsilon \text{ for } r_0 > R.$$

Now from the above it follows that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^k}=0.$$

Thus the lemma is established. ∎

Lemma 3 *[2]If f is a non constant entire function then*

$$T(r,f)\le\log M(r,f)\le\log T(2r,f)+o(1)\ \text{as}\ r\to\infty.$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be meromorphic with L^*-order zero. Also let $0<\rho_f^{*L^*}<\infty$. Then Definition A and Definition B are equivalent.*

Proof. Case I: $\sigma_f^{*L^*}=\infty$.

Definition A $\Rightarrow$ Definition B.

As $\sigma_f^{*L^*}=\infty$, from Definition A we obtain for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\begin{aligned} T(r,f) &> G\log\{r\exp(L(r))\}^{\rho_f^{*L^*}} \\ \text{i.e.,}\qquad \exp(T(r,f)) &> \left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^G. \end{aligned}\tag{1}$$

If possible, let the integral

$$\int_{r_0}^{\infty}\frac{\exp[T(r,f)]\,dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{G+1}}\quad(r_0>0)$$

be converge. Then by Lemma 1 we get that

$$\limsup_{r\to\infty}\frac{\exp[T(r,f)]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{G}}=0.$$

So for all sufficiently large values of r,

$$\exp[T(r,f)]<\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^G.\tag{2}$$

Now from (1) and (2) we arrive at a contradiction. Hence

$$\int_{r_0}^{\infty}\frac{\exp[T(r,f)]\,dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{G+1}}\quad(r_0>0)$$

diverges whenever G is finite, which is Definition B.

Definition B $\Rightarrow$ Definition A.

Let G be any positive number. Since $\sigma_f^{*L^*} = \infty$, from Definition B the divergence of the integral,

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{G+1}} \quad (r_0 > 0)$$

gives for arbitrary positive ϵ and for a sequance of values of r tending to infinity

$$\exp\left[T(r,f)\right] > \left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{G-\epsilon}$$

$$\text{i.e.,} \quad T(r,f) > (G-\epsilon)\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}.$$

This gives that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} \geq G-\epsilon$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} = \infty.$$

Thus Definition A follows.

Case II: $0 \leq \sigma_f^{*L^*} < \infty$.

Definition A $\Rightarrow$ Definition B

Subcase (a): Let f be of L*-type $\sigma_f^{*L^*}$ where $0 < \sigma_f^{*L^*} < \infty$. Then for arbitrary $\epsilon > 0$ and for all sufficiently large values of r,

$$\frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} < \sigma_f^{*L^*} + \epsilon$$

$$\textit{i.e.,} \quad T(r,f) < (\sigma_f^{*L^*}+\epsilon)\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}$$

$$\textit{i.e.,} \quad \exp\left[T(r,f)\right] < \exp\left[(\sigma_f^{*L^*}+\epsilon)(\log re^{L(r)})^{\rho_f^{*L^*}}\right]$$

$$\textit{i.e.,} \quad \exp\left[T(r,f)\right] < \left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{(\sigma_f^{*L^*}+\epsilon)}$$

$$\textit{i.e.,} \quad \frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k}} < \frac{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{(\sigma_f^{*L^*}+\epsilon)}}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k}}$$

$$\textit{i.e.,} \quad \frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k}} < \frac{1}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k-(\sigma_f^{*L^*}+\epsilon)}}.$$

Therefore
$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} \quad (r_0 > 0)$$
converges if $k > \sigma_f^{*L^*}$ and diverges if $k < \sigma_f^{*L^*}$.

Subcase (b): When f is of type $\sigma_f^{*L^*} = 0$.

Definition A gives for all sufficiently large values of r that
$$\frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} < \epsilon.$$
Then as before we obtain that
$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}} \quad (r_0 > 0)$$
converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (a) and Subcase (b), Definition B follows.

Definition B $\Rightarrow$ Definition A

Since f be of L*-type $\sigma_f^{*L^*}$, by Definition B for arbitrary $\epsilon\,(> 0)$ the integral
$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1+\epsilon}}$$
converges. Then by Lemma 1
$$\limsup_{r\to\infty} \frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+\epsilon}} = 0$$
i.e., for all sufficiently large values of r,
$$\frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+\epsilon}} < \epsilon$$
$$i.e., \qquad \exp\left[T(r,f)\right] < \epsilon\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+\epsilon}$$
$$i.e., \qquad T(r,f) < \log\epsilon + \left(\sigma_f^{*L^*}+\epsilon\right)\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}$$
$$i.e., \qquad \frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} < \frac{\log\epsilon}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} + \left(\sigma_f^{*L^*}+\epsilon\right)$$
$$i.e., \qquad \limsup_{r\to\infty}\frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}} \le \sigma_f^{*L^*}+\epsilon.$$

Since $\epsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}}\le\sigma_f^{*L^*}. \tag{3}$$

Again by Definition B, the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1-\epsilon}}$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1-\epsilon}}>\frac{1}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{1+\epsilon}}$$

$$i.e.,\qquad \exp\left[T(r,f)\right]>\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}-2\epsilon}$$

$$i.e.,\qquad T(r,f)>\left(\sigma_f^{*L^*}-2\epsilon\right)\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}$$

$$i.e.,\qquad \frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}}>\sigma_f^{*L^*}-2\epsilon.$$

As $\epsilon\,(>0)$ is arbitrary we get that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}}\ge\sigma_f^{*L^*}. \tag{4}$$

Therefore from (3) and (4) it follows that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left[\log\left\{re^{L(r)}\right\}\right]^{\rho_f^{*L^*}}}=\sigma_f^{*L^*}.$$

Thus we obtain Definition A.
Now combining Case I and Case II, the theorem follows. ■

Theorem 2 *The integral*

$$\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\quad (r_0>0)$$

follows if and only if the integral

$$\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\quad \textit{converges.}$$

Proof. *Let*

$$\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\quad (r_0>0)$$

converges. Then by the first part of Lemma 3, we obtain that

$$\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\leq\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}$$

i.e.,

$$\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\text{ converges.}$$

Next let

$$\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\quad (r_0>0)$$

be convergent. Then by the second part of Lemma 3, we get that

$$\begin{aligned}&\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\\ <\ &\int_{r_0}^{\infty}\frac{\exp\left[T(2r,f)\right]dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}+\int_{r_0}^{\infty}\frac{o(1)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}\\ =\ &\frac{1}{2\left[\exp\left(\frac{1}{2}\rho_f^{*L^*}\right)\right]}\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(re^{\left(\frac{r}{2}\right)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}+o(1).\end{aligned}$$

Thus

$$\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{\sigma_f^{*L^*}+1}}$$

converges. This proves the theorem. ■

Now in view of Theorem 1 and Theorem 2 we may give an alternative definition of L*-type $\sigma_f^{*L^*}$ of an entire function f with L*-order zero as follows:

An entire function f with L*-order zero is said to be of type $\sigma_f^{*L^*}$ if the integral

$$\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\left\{\log\left(re^{L(r)}\right)\right\}^{\rho_f^{*L^*}}\right]^{k+1}}\quad (r_0>0)$$

converges for $k>\sigma_f^{*L^*}$ and diverges for $k<\sigma_f^{*L^*}$.

Theorem 3 *If f be a meromorphic function of infinite L^*-order and $0 < \overline{\rho}_f^{L^*} < \infty$ then Definition C and Definition D are equivalant.*

Proof. Case I: $\overline{\sigma}_f^{L^*} = \infty$.

Definition C$\Rightarrow$ Definition D

As $\overline{\sigma}_f^{L^*} = \infty$, from Definition C, we obtain for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T(r,f) &> G\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}} \\ i.e., \quad T(r,f) &> \left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^G. \end{aligned} \tag{5}$$

If possible, let the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{G+1}} \quad (r_0 > 0)$$

be convergent. Then by Lemma 2

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{G}} = 0.$$

So for all sufficiently large values of r,

$$T(r,f) < \left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^G. \tag{6}$$

Now from (5) and (6) we arrive at a contradiction. Hence

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{G+1}} \quad (r_0 > 0)$$

diverges whenever G is finite, which is Definition D.

Definition D$\Rightarrow$ Definition C

Let G be any positive number. Since $\overline{\sigma}_f^{L^*} = \infty$, from Definition D the divergence of the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{G+1}} \quad (r_0 > 0)$$

gives for arbitrary positive ϵ and for a sequence of values of r tending to infinity,

$$\begin{aligned} T(r,f) &> \left[\exp\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]^{G-\epsilon} \\ i.e., \log T(r,f) &> (G-\epsilon)\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}. \end{aligned}$$

This gives that

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}}\geq G-\epsilon.$$

Since G is arbitrary, this shows that

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}}=\infty.$$

Thus Definition C follows.

Case II: $0\leq\overline{\sigma}_f^{L^*}<\infty$.

Definition C$\Rightarrow$ Definition D.

Subcase(a): Let f be of L*-type $\overline{\sigma}_f^{L^*}$ where $0\leq\overline{\sigma}_f^{L^*}<\infty$, Then according to Definition C, for arbitrary positive ϵ and for all sufficiently large values of r we get that

$$\log T(r,f)<\left(\overline{\sigma}_f^{L^*}+\epsilon\right)\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}$$

$$i.e.,\qquad T(r,f)<\exp\left[\left(\overline{\sigma}_f^{L^*}+\epsilon\right)\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]$$

$$i.e.,\qquad T(r,f)<\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\left(\overline{\sigma}_f^{L^*}+\epsilon\right)}$$

$$i.e.,\qquad \frac{T(r,f)}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'}}<\frac{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\left(\overline{\sigma}_f^{L^*}+\epsilon\right)}}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'}}$$

$$i.e.,\qquad \frac{T(r,f)}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'}}<\frac{1}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'-\left(\overline{\sigma}_f^{L^*}+\epsilon\right)}}.$$

Therefore

$$\int_{r_0}^{\infty}\frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'}}\qquad(r_0>0)$$

converges if $k'>\overline{\sigma}_f^{L^*}$ and diverges if $k'<\overline{\sigma}_f^{L^*}$

$$i.e.,\ \int_{r_0}^{\infty}\frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'+1}}\qquad(r_0>0)$$

converges if $k'>\overline{\sigma}_f^{L^*}$ and diverges if $k'<\overline{\sigma}_f^{L^*}$.

Subcase (b): When f is of L*-type $\overline{\sigma}_f^{L^*} = 0$, Definition C gives for all sufficiently large values of r,

$$\frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} < \epsilon.$$

Then as before we obtain that

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k'+1}} \quad (r_0 > 0)$$

converges if $k' > 0$ and diverges if $k' < 0$. Thus combining Subcase (a) and Subcase (b), Definition D follows.

Definition D$\Rightarrow$ Definition C.

Since f is of L*-type $\overline{\sigma}_f^{L^*}$, by Definition D, for arbitrary $\epsilon(>0)$ the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1+\epsilon}}$$

converges. Then by Lemma 2, we obtain that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+\epsilon}} = 0$$

i.e. for all sufficiently large values of r,

$$\frac{T(r,f)}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+\epsilon}} < \epsilon$$

$$i.e., \quad T(r,f) < \epsilon\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+\epsilon}$$

$$i.e., \quad \log T(r,f) < \log\epsilon + \left(\overline{\sigma}_f^{L^*}+\epsilon\right)\left[\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}\right]$$

$$i.e., \quad \frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} < \frac{\log\epsilon}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} + (\overline{\sigma}_f^{L^*}+\epsilon).$$

Since $\epsilon\,(>0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} \le \overline{\sigma}_f^{L^*}. \qquad (7)$$

Again by Definition D, for arbitrary positive ϵ, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1-\epsilon}}$$

implies that there exist a sequence of values of r tending to infinity such that

$$\frac{T(r,f)}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1-\epsilon}} > \frac{1}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{1+\epsilon}}$$

$$i.e., \quad T(r,f) > \left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}-2\epsilon}$$

$$i.e., \quad \log T(r,f) > \left(\overline{\sigma}_f^{L^*} - 2\epsilon\right)\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}$$

$$i.e., \quad \frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} \geq \overline{\sigma}_f^{L^*} - 2\epsilon$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} \geq \overline{\sigma}_f^{L^*} - 2\epsilon.$$

As $\epsilon\,(>0)$ is arbitrary we obtain from above that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} \geq \overline{\sigma}_f^{L^*}. \tag{8}$$

Now from (7) and (8) it follows that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\left\{re^{L(r)}\right\}^{\overline{\rho}_f^{L^*}}} = \overline{\sigma}_f^{L^*}.$$

Thus we get Definition C.
Hence combining Case I and Case II, the theorem follows. ■

Theorem 4 *The integral*

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \quad (r_0 > 0)$$

converges if and only if the integral

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \quad (r_0 > 0) \text{ converges.}$$

Proof. Let

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \quad (r_0 > 0)$$

be convergent . Then by the first part of Lemma 3, we obtain that

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \leq \int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}}$$

$$i.e., \quad \int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \quad (r_0 > 0) \ \ converges.$$

Next let

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \quad (r_0 > 0)$$

be convergent. Then by the second part of Lemma 3, we get that

$$\begin{aligned} & \int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \\ \leq & \int_{r_0}^{\infty} \frac{T(2r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} + \int_{r_0}^{\infty} \frac{o(1)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \\ = & \frac{1}{2\left[\exp\left(\frac{1}{2}\overline{\rho}_f^{L^*}\right)\right]} \int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(re^{L\left(\frac{r}{2}\right)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} + o(1). \end{aligned}$$

Thus

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{\overline{\sigma}_f^{L^*}+1}} \quad (r_0 > 0)$$

is convergent.
This proves the theorem. ■

Now in view of Theorem 3 and Theorem 4, we may give an alternative deifnition of L*-type $\overline{\sigma}_f^{L^*}$ of an entire function f with infinite L*-order as follows:

An entire function f with L*-infinite order is said to be of L*-type $\overline{\sigma}_f^{L^*}$ if the integral

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\left\{\left(re^{L(r)}\right)^{\overline{\rho}_f^{L^*}}\right\}\right]^{k+1}} \quad (r_0 > 0)$$

converges for $k > \overline{\sigma}_f^{L^*}$ and diverges for $k < \overline{\sigma}_f^{L^*}$.

References

[1] Hayman, W.K.: Meromorphic functions, The Clarendon Press, Oxford (1964).

[2] Lahiri, I.: Generalised order of meromorphic functions, Soochow J. Math., Vol. 17, No.1 (1991), pp.67-82.

[3] Liao, L. and Yang, C.C.: On the growth of composite entire functions, Yokohama Math. J., Vol. 46 (1999), pp. 97-107.

[4] Somasundaram, D. and Thamizharasi, R.: A note on the entire functions of L-bounded index and L-type, Indian J. Pure and Appl. Math., Vol. 19, No. 3 (March 1988), pp. 284-293.

[5] Valiron, G.: Lectures on the general theory of integral functions, Chelsea Publishing Company (1949).