On The Numerical Approximation of Volterra Integral Equations of Second kind Using Quadrature Rules

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Repeated Simpson's and Trapezoidal quadrature rule was used to solve the linear Volterra Integral Equations of the second kind. A recurrence relation was derived and an approximation solution is obtained. We show that our estimates have a good degree of accuracy.

Keywords: Linear Volterra Integral Equations of the Second kind, Repeated Simpson's, Trapezoidal rule, quadrature rule, Numerical analysis.

Introduction

The problem of finding numerical solution for Fredholm and Volterra integral equations of the second kinds is one of the oldest problems in the applied Mathematics literature and many computational methods are introduced in this field ([1,6,7]). Many methods discuss about the linear Fredholm integral equation (see [6]). In ([12]), Saberi-Nadja and Heidari by applying modified trapezoid formula, solved linear Frdholm integral equation . Many approaches for solving the linear and nonlinear kind of these equation may be found in [2,3,4,8,10]. A numerical quadrature rules is a primary tool used by engineers and scientists to obtain approximate answers for definite integrals that cannot be solved analytically. Also, it is a basis of every numerical method for solution of integral equation. Using quadrature methods, we will solve Volterra Integral Equations of the Second kind. We take the advantage of the linearity of the Volterra Integral Equations .We can use a numerical method of the evaluation of integral . we will consider two methods , the first is the Trapezoidal rule and the second is the Simpson's rule We first present the most familiar of numerical integration the quadrature rules (Trapezoidal rule, Simpson's l/3 rule). Then a linear system must be solved. In our study will investigate the case of a linear integral equation

$$\varphi(x) - \lambda \int_a^b K(x,y)\psi(y)dy = f(x) \qquad (1)$$

$$\int_a^b f(x)dx = \sum_{k=1}^{n} w_k f(x_k) + E(f)$$

is the numerical integration on quadrature formula x_k: are the nodes w_k: are the weights , and it is not depend on f(x) , φ: unknown function and f: known function

2.Trapezoidal rule

Let $a < b \in \mathbb{R}$, we divide the interval (a, b) into subinterval with equal length $h = \frac{b-a}{N}$, $N \in \mathbb{N}^*$, we denote $x_k = a + (k-1)h$, $1 \le k \le N+1$.Then, the Trapezoidal method says :

$$\int_a^b f(x)dx = h\left[\frac{f(a) + f(b)}{2} + \sum_{k=2}^{N-1} f(x_k)\right] \qquad (2)$$

Using the Trapezoidal approximation to solve the Volterra Integral Equations:

$$\varphi(x) - \lambda \int_a^x K(x, y)\psi(y)dy = f(x) \qquad (3)$$

We substitute (2) into the Volterra integral equations with $x = x_i$, we get

$$\varphi(x_i) - h\left[\frac{K(x_i, a)\varphi(a) + K(x_i, x_i)\varphi(x_i)}{2} + \sum_{j=2}^{i-1} K(x_i, x_j)\varphi(x_j)\right] = f(x_i)$$

$$1 \le i \le N+1, \qquad x_1 = a, \dots, x_{N+1} = b$$

$$-h\frac{K(x_i, a)}{2}\varphi(a) - h\sum_{j=2}^{i-1} K(x_i, x_i)\varphi(x_j) + \left(1 - h\frac{K(x_i, x_i)}{2}\right)\varphi(x_i) = f(x_i)$$

For $i = 1$, $x_1 = a$, using (3) the Volterra Integral Equations is reduced to

$$\varphi(a) = f(a)$$

For $i = 2$, we have

$$-h\frac{K(x_2, x_1)}{2}\varphi(x_1) + \left[1 - h\frac{K(x_2, x_2)}{2}\right]\varphi(x_2) = f(x_2)$$

For $i = 3$

$$-h\frac{K(x_3, x_1)}{2}\varphi(x_1) - h\,K(x_3, x_2)\varphi(x_2) + \left[1 - h\frac{K(x_3, x_3)}{2}\right]\varphi(x_3) = f(x_3)$$

Now we get a linear system which will be solved

$$A\overline{\varphi} = \overline{b}$$

With

$$\overline{b} = [f(x_1) = f(a), f(x_2), \dots, f(x_{N+1}) = f(b)]^T , \ \overline{\varphi} = [\varphi(a), \varphi(x_2), \dots, \varphi(x_{N+1})]^T$$

Where "T" indicate the transpose of vector , and the matrix $A = (a_{ij})$, $1 \le i$, $j \le N+1$ with :

$$\begin{cases} a_{ij} = 0, & \forall \quad j > i+1 \\ a_{ij} = -h\,K(x_i, x_j), 2 \le j < i \le n+1 \\ a_{ii} = 1 - \frac{h}{2} K(x_i, x_i) \\ a_{11} = 1 \\ a_{i1} = -\frac{h}{2} K(x_i, x_1), 1 \le i \le n+1 \end{cases}$$

Then

$$A = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & & 0 \\ & \vdots & & \ddots & \vdots \\ a_{N+1,1} & a_{N+1,2} & \cdots & \cdots & a_{N+1,N+1} \end{pmatrix}$$

Following the previous method , we present in the next section Simpson's methods .

3. Simpson's methods

In all our approximation, the error assumed negligible. It's follows that at the r^{th} stage if the number of subintervals are even we apply the Simpson's rule. If N is even then Simpson's quadrature rule may be applied to each subinterval $[x_{2i}, x_{2i+1}, x_{2i+2}]$. For $i = 0, \dots, \frac{N}{2} - 1$ we have

$$\int_{x_{2i}}^{x_{2i+2}} f(x)dx \simeq \frac{h}{3}[\, f(x_{2i}) + 4\, f(x_{2i+1}) + f(x_{2i+2})\,]$$

Summing up :

$$\int_a^b f(x)dx = \sum_{k=0}^{N-1} \int_{x_{2k}}^{x_{2k+2}} f(x)dx = \sum_{k=0}^{N-1} \frac{h}{3}\left[f(x_{2k}) + 4\, f\left(\frac{x_{2k} + x_{2k+2}}{2}\right) + f(x_{2k+2})\right]$$

$$= \frac{h}{3} \sum_{k=0}^{N-1} \left[f(x_{2k}) + 4\, f\left(\frac{x_{2k} + x_{2k+2}}{2}\right) + f(x_{2k+2})\right]$$

We will use this approximation to solve Volterra Integral Equations of the Second kind :

$$\varphi(x) - \lambda \int_a^x K(x, y)\psi(y)dy = f(x)$$

The approximation of Volterra Integral Equations in the even nodes is given by

$$\varphi(x_{2m}) - \int_a^{x_{2m}} K(x_{2m}, y)\varphi(y)dy = f(x_{2m})$$

For simplicity we denote

$$f(x_{2m}) = f_{2m} \;, \qquad \varphi(x_{2m}) = \varphi_{2m}$$

Volterra Integral Equations of the Second Kind becomes :

$$\varphi(x_{2m}) - \frac{h}{3}\sum_{k=0}^{m-1}\left[K(x_{2m}, y_{2k})\varphi_{2k} + 4K\left(x_{2m}, \frac{y_{2k}+y_{2k+2}}{2}\right)\right] + K(x_{2m}, y_{2k+2})\varphi(x_{2k+2}) = f(x_{2m})$$

$$\varphi_{2m} - \frac{h}{3}\sum_{k=0}^{m-1} K_{2m,2k}\ \varphi_{2k} + 4K_{2m,2k+1}\ \varphi_{2k+1} + K_{2m,2k+2}\ \varphi_{2k+2} = f_{2m}$$

Using this approximation:

$$\varphi_{2k+1} \simeq \frac{\varphi_{2k} + \varphi_{2k+2}}{2}$$

Then we get

$$\varphi_{2m} - \frac{h}{3}\sum_{k=0}^{m-1} K_{2m,2k}\varphi_{2k} + 4K_{2m,2k+1}\frac{\varphi_{2k}+\varphi_{2k+2}}{2} + K_{2m,2k+2}\ \varphi_{2m+2} = f_{2m}$$

$$\varphi_{2m} - \frac{h}{3}\sum_{k=0}^{m-1}\left(K_{2m,2k} + 2K_{2m,2k+1}\right)\varphi_{2k} + \left(K_{2m,2k+2} + 2K_{2m,2k+1}\right)\varphi_{2k+2} = f_{2m}$$

$$\varphi_{2m} - \frac{h}{3}\left\{\sum_{\boldsymbol{k=0}}^{\boldsymbol{m-1}}(K_{2m,2k} + 2K_{2m,2k+1})\ \varphi_{2k} + \sum_{k=1}^{m}(K_{2m,2k} + 2K_{2m,2k-1})\varphi_{2k}\right\} = f_{2m}$$

$$\begin{aligned}\varphi_{2m} - \frac{h}{3}&\left[(K_{2m,0} + 2K_{2m,1})\right]\varphi_0 + \left(K_{2m,2m} + 2K_{2m,2m-1}\right)\varphi_{2m}\\ &- \frac{h}{3}\sum_{k=1}^{m-1}\left(\ 2\,K_{2m,2k} + 2\,K_{2m,2k+1} + 2K_{2m,2k-1}\ \right)\varphi_{2k} = f_{2m}\end{aligned}$$

$$\begin{aligned}\varphi_{2m} - \frac{h}{3}\left[(K_{2m,0} + 2K_{2m,1})\varphi_0\right] - \frac{h}{3}\left[K_{2m,2m} + 2K_{2m,2m-1}\right]\varphi_{2n} - \frac{2h}{3}\sum_{k=1}^{m-1}(K_{2m,2k-1} + K_{2m,2k}\\ + K_{2m,2k+1})\ \varphi_{2k} = f_{2m}\end{aligned}$$

$$\left(1 - \frac{h}{3}\left[K_{2m,2m} + 2K_{2m,2m-1}\right]\right) = f_{2m} + \frac{h}{3}(K_{2m,0} + 2K_{2m,1})\varphi_0 + \frac{2h}{3}\sum_{k=1}^{m-1} c_k\ \varphi_{2k}$$

where $c_k = K_{2m,2k-1}K_{2m,2k} + K_{2m,2k+1}$

if

$$1 - \frac{h}{3}\left[K_{2m,2m} + 2K_{2m,2m-1}\right] \neq 0$$

then we get a linear system must be solved

$$\varphi_{2m} = \frac{f_{2m} + \frac{h}{3}\left(K_{2m,0} + 2K_{2m,1}\right)\varphi_0 + \frac{2\,h}{3}\sum_{k=1}^{m-1} c_k \varphi_{2k}}{1 - \frac{h}{3}\left(K_{2m,2m} + 2K_{2m,2m-1}\right)} \quad , m = 1,2.., \frac{N}{2}$$

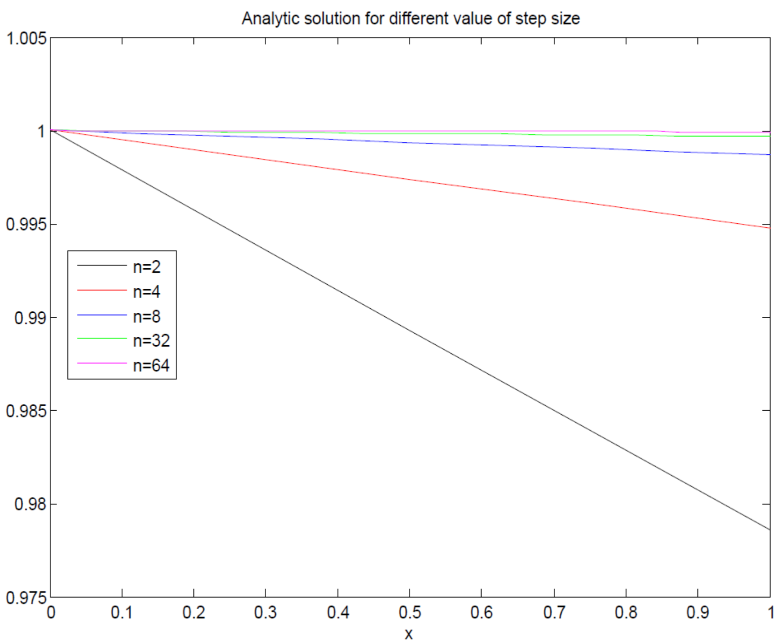
$$\varphi(a) = \varphi_0 = f(a)$$

4. Numerical Example

In this section, to achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results. The computations, associated with the example, are performed by MATLAB 7.

Now, the accuracy of the solution depends on the step-size. As an illustration, we solve the same problem that we solved analytically in the last section. This time we solve it with this method for various values of the step-size. This example gives the numeric solution to Volterra Equations with $a = 0, \lambda = -1, f(x) = e^x$ and $K(x, y) = e^{x-y}$.

Then we will present different cases, where we change the step size just to see the effect, this is shown in Figure 1. We will take these case with: $n = \{2,4,8,32,64\}$. Clearly as the number of intervals, n , increases, the accuracy of our solution to the integral equation also increases, such a result that should not surprise as.



Example: We look to the linear Volterra Integral Equations of the second kind:

$$\varphi(x) - \int_0^x e^{y-x}\,\varphi(y)dy = f(x)\,, \quad 0 \le x \le 1$$

Exact solution $\varphi(x) = x$, with $f(x) = 1 - e^{-x}$, $\lambda = 1$

The Numerical and exact solution are compared using the absolute error $|\varphi(x) - \varphi_n(x)|$

h x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
h=0.1	0	$5\ 10^{-6}$	$5.6\ 10^{-7}$	$8.3\ 10^{-8}$	$7\ 10^{-8}$	$2.2\ 10^{-9}$	$5\ 10^{-9}$	$4\ 10^{-9}$	$5\ 10^{-10}$	$7\ 10^{-10}$
h=10^{-2}	0	$7\ 10^{-7}$	$4.1\ 10^{-9}$	$5\ 10^{-10}$	$7\ 10^{-11}$	$2\ 10^{-11}$	$6.92\ 10^{-12}$	$4.3\ 10^{-12}$	$5.3\ 10^{-13}$	$2.1\ 10^{-14}$
h=10^{-3}	0	$2\ 10^{-9}$	$1.19\ 10^{-10}$	$4.2\ 10^{-11}$	$5.13\ 10^{-}$	$7.1\ 10^{-12}$	$5.3\ 10^{-14}$	$3.4\ 10^{-14}$	$5\ 10^{-15}$	$2\ 10^{-16}$

Some Remarks must be made :

- The error depend on the parameter "h" and it's decreasing according to h
- We divide the interval of integration $[0, x]$ into "m" equal subintervals with equal length .
- Clearly as the number of intervals increased the accuracy of solution of the integral equation also increases.

5. Conclusion

We applied repeated Simpson's quadrature rule and Trapezoidal for solving the linear Volterra Integral Equations of the Second kind. According to the numerical results which obtaining from the illustrative examples, we conclude that for sufficiently small h we get a good accuracy. The same approach can be used to solve other problems like:

- Nonlinear Volterra integral equations.
- Linear Fredholm integral equations with Cauchy kernel , Abel kernel.
- Voltterra- Fredholm integral equations.

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