An Investigation of the Performance of Kolmogorov-Smirnov and Anderson-Darling Goodness of Fit Statistics on Pareto and Gumbel Minimum Densities

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Abstract

The study investigated agreement between Kolmogorov-Smirnov and Anderson-Darling goodness of fit statistics on simulated Pareto and Gumbel Minimum densities. The simulation was carried out with EasyFit Version 4.3 software and 36 continuous distributions were examined for fit on each simulated distribution. Sample sizes 20, 100 and 1000 were used and 100 replications performed. The observed coefficients were also modeled. Results revealed: supremacy of Komogorov-Smirnov test over Anderson-Darling test for Pareto (for all sample sizes) and supremacy of Anderson-Darling test over Kolmogorov-Smirnov test for Gumbel Minimum (for small and moderate sample sizes); stalemate between Kolmogorov-Smirnov and Anderson-Darling for large n (n=1000) in the case of Gumbel Minimum; improved agreement between Kolmogorov-Smirnov and Anderson-Darling with increasing sample size for case of Pareto; constant agreement between Kolmogorov-Smirnov and Anderson-Darling for all sample sizes in the case of Gumbel Minimum. Recommendation for inclusion of more distributions and goodness of fit tests was made

Keywords: Fit; Pareto; Gumbel Minimum; Kolmogorov-Smirnov; Anderson-Darling; Simulation.

1.0 INTRODUCTION

Many a time, the need to test whether a set of observations follow a particular distribution or whether two or more sets follow the same distribution arises. Such an exercise may be informed by the need to satisfy distributional assumptions of a certain statistical procedure or the need to associate that particular type of data with a particular distribution or family of distributions. The test in question called 'the goodness of fit' test is traditionally investigated by the chi-square statistic due to Pearson (1900). After this, a lot of other statistics have been developed.

These include the likelihood ratio statistic due to Wilks (1938), Neyman modified chi-square due to Neyman (1949), Freeman-Tukey statistic due to Freeman and Tukey (1950), Anderson-Darling due to Anderson and Darling (1952), Kuper's test due to Kuper (1962), Shapiro-Wilks test due to Shapiro and Wilks (1965), deviance method due to Nelder and Wedderburn (1972), Jarque-Bera test due to Jarque and Bera (1980), Cressie-Read statistic due to Cressie and Read (1984), and Kolmogorov-Smirnov test. Efforts at comparing tests include West and Kempthorne (1972), Larntz (1978), Lawal (1984, 1989, 1993)), Lawal and Upton (1990).

The aim of this article is to investigate agreement between Kolmogorov-Smirnov and Anderson-Darling tests on simulated Pareto and Gumbel Minimum densities.

The article is organized as follows: Section 2 presents the methods; section 3 presents the results and discussions while the last section presents the conclusion.

1.1Kolmogorov-Smirnov (K-S) Test

Kolmogorov-Smirnov (K-S) tests whether or not a given distribution is not significantly different from hypothesized distribution. The test is based on the Empirical Cumulative Distribution Function (ECDF).

Let's assume that we have a random sample (of size n) $x_1, x_2, \ldots$, xn from some pdf with cumulative distribution function (CDF) F(x), then the ECDF is denoted:

$$F_n(x) = \frac{1}{n}[\text{Number of observations} \leq \text{x}] \qquad (1.1.1a)$$

The K-S statistic (D) is based on the largest vertical difference between the theoretical and the empirical CDF and is defined:

$$D = \max_{1 \le i \le n}[F(x_i) - \tfrac{i-1}{n}, \tfrac{\text{i}}{\text{n}} - F(x_i)] \qquad (1.1.1b)$$

The null and the alternative hypotheses are:

H_0: The data follow the specified distribution

H_1: The data do not follow the specified distribution

H_0 is rejected if D > critical value or if p-value < α

1.2 Anderson-Darling (A-D) Test

The Anderson-Darling (A) test, due to Anderson and Darling (1952) is a general test for comparing the fit of an empirical CDF to a theoretical CDF. It is in the literature that this test assigns more weight to the tails than does the K-S test.

A-D test statistic (A^2) is defined as:

$$A^2 = -n - \tfrac{1}{n}\sum(2i-1)[InF(x_i) + In(1-F(x_i))] \qquad (1.2.1)$$

The null and the alternative hypotheses are same as those of K-S test.

The null hypothesis is rejected if A^2 > critical value. The critical values for this procedure depend on the specific distribution being tested and this poses a great problem, particularly if the distribution in question is not one of the most widely used ones, whose tables are available. However an approximation formula which depends only on the sample size has been developed.

1.3 Pareto Distribution

The Pareto distribution is a positively skewed continuous distribution originally used for modeling allocation of wealth among individuals. It describes a situation where the larger portion of the wealth of the society is owned by a smaller percentage of the people- a typical situation in any society. Frequencies of words in longer texts, size of human settlements, size of sand particles are cases that can be modeled by Pareto. This distribution is often described in relation to '80/20' Rule. In economics, it may mean that 20% of the inputs create 80% of the result; 20% of the customers create 80% of the revenue; 20% of the population hold 80% of the wealth. The basic principle is that most things in life are not evenly distributed.

The Pareto distribution has two parameters (α and β) and is defined as:

$$f(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}} \qquad \beta \le x < \infty;\ \alpha > 0;\ \beta > 0 \tag{1.3.1}$$

α and β are shape and scale parameters respectively

$$F(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha} \tag{1.3.2}$$

$$E(X) = \frac{\alpha\beta}{\alpha - 1} \tag{1.3.3a}$$

$$V(X) = \left(\frac{\beta}{\alpha - 1}\right)^2 \left(\frac{\alpha}{\alpha - 2}\right) \tag{1.3.3b}$$

The implication is that E(X) does not exist if $\alpha \le 1$ and V(X) does not exist if $\alpha \le 2$.

$$\mu_n^{/} = \frac{\alpha\beta^n}{\alpha - n} \qquad \alpha > n \tag{1.3.3c}$$

$$Skewness = 2\sqrt{\frac{\alpha - 2}{2}}\,\frac{(\alpha + 1)}{(\alpha - 3)} \qquad \alpha > 3 \tag{1.3.3d}$$

$$Kurtosis = \frac{3(\alpha - 2)(3\alpha^3 + \alpha + 2)}{\alpha(\alpha - 3)(\alpha - 4)} \qquad \alpha > 4 \tag{1.3.3e}$$

Hence raw moments, variance, skewness and the kurtosis are defined but not for all values of the shape parameter, α.

Random number generation can be performed for a Pareto distribution by transforming a continuous uniform variable U(0, 1) with the distribution's inverse function as follows:

$$Par\,(\alpha, \beta) = \beta\left(\frac{1}{1\text{-}\,\mathrm{U}(0,1)}\right)^{\frac{1}{\alpha}} \qquad (1.3.4)$$

1.4 Gumbel Distribution

The Gumbel distribution is used for modeling the distribution of the maximum (minimum) of a number of samples of organizations. It may be used to model the maximum (minimum) water level of a river. It is particular case of generalized extreme value distribution (otherwise known as Fisher-Tippet distribution). Their exist two types of Gumbel distribution namely: Gumbel minimum and Gumbel maximum. The former is of particular interest to this article. The Gumbel minimum distribution (also called log-Weibull distribution or double exponential distribution) is negatively skewed and has two parameters (α and β).

The pdf is:

$$f(x) = \frac{1}{\beta}\exp\left(\frac{x-\alpha}{\beta} - \exp\left(\frac{x-\alpha}{\beta}\right)\right) \qquad -\infty < x < \infty \quad (1.4.1)$$

$$\beta > 0$$

α and β are scale and location parameters respectively.

$$F(x) = 1 - \exp\left(-\exp\left[\frac{x-\alpha}{\beta}\right]\right) \quad (1.4.2)$$

$$E(X) = \alpha + .5772\beta \quad (1.4.3a)$$

$$V(X) = \tfrac{1}{6}\Pi^2\beta^2 \quad (1.4.3b)$$

$$Skewness = 1.13955 \quad (1.4.3c)$$

$$Kurtosis = 5.4 \quad (1.4.3d)$$

When $\alpha = 0$ and $\beta = 1$ in (1.4.1), we have the standard Gumbel minimum defined as:

$$f(x) = \exp[x - \exp(x)] \quad (1.4.4)$$

(1.4.4) is of particular interest to this research.

Gumbel random numbers can be generated by transforming a continuous Uniform variable, U(0, 1) with the distribution's inverse probability function as follows:

$$G(\alpha, \beta) = \alpha - \beta\, \text{In}\left(\text{In}\frac{1}{1 - \text{U}(0,1)}\right) \quad (1.4.5)$$

2.0 METHODS

2.1 Design of Experiments

Each of Pareto (with parameters $\alpha = 5$ and $\beta = 1$) and standard Gumbel minimum distributions was simulated for sample sizes 20, 100, and 1000. Kolmogorov-Smirnov and Anderson-Darling tests were applied on the simulated data for fit on 36 continuous distributions, which were ranked on the order of goodness of fit to the simulated data. Spearman's correlation coefficient (r) was observed between the ranks assigned by the two statistics. Median r was obtained for each scenario (distribution and sample size). The observed r values were also modeled for each scenario. 100 replications were performed. EasyFit 4.3 and SPSS 17.0 were used for the simulation and analysis.

2.2 Model Estimation

Model estimation was carried out by the maximum likelihood method (MLE).

Maximum Likelihood Estimation of Pareto parameters

$$f(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}$$

The likelihood function L is:

$$L = \frac{\alpha^{n}\beta^{n\alpha}}{\left(\prod_{i=1}^{n} x_i\right)^{\alpha+1}} \qquad (2.2a)$$

Taking log

$$\log \mathrm{L} = \mathrm{n}\log\alpha + \mathrm{n}\alpha\log\beta - (\alpha+1)\log\prod x \tag{2.2b}$$

$$\frac{\mathrm{dlog\ L}}{\mathrm{d}\alpha} = \frac{n}{\alpha} + n\log\beta - \sum\log x = 0 \tag{2.2c}$$

$$\hat{\alpha} = \frac{\mathrm{n}}{\sum\log\frac{\mathrm{x}}{\hat{\beta}}} \tag{2.2d}$$

MLE of β can not be found by differentiation as done for α since logL is unbounded for β. Noting that β is the lower bound of x (see (1.3.1)), logL can be maximized subject to the constraint:

$$\hat{\beta} \leq \min_i x_i \tag{2.2e}$$

log L is maximized with respect to β when

$$\hat{\beta} = \min_i x_i \tag{2.2f}$$

(2.2f) is hence the MLE of β. Saksena and Johnson (1984) have shown that MLE of α and β are best linear unbiased estimators.

MLE of Gumbel Minimum Parameters

$$f(x) = \frac{1}{\beta}\exp\left(\frac{x-\alpha}{\beta} - \exp\left(\frac{x-\alpha}{\beta}\right)\right) \tag{2.2g}$$

$$L = \left(\frac{1}{\beta}\right)^n \exp\left\{-\sum_{i=1}^{n}\left(\left[\frac{x_i-\alpha}{\beta} + \exp\left(-\frac{x_i-\alpha}{\beta}\right)\right]\right)\right\} \tag{2.2f}$$

$$= \frac{n}{\beta} - \frac{1}{\beta} \sum \exp\left(- \frac{x_i - \alpha}{\beta} \right) \quad \log L = -n \log \beta - \sum \left\{ \left(\frac{x_i - \alpha}{\beta} + \exp\left(- \frac{x_i - \alpha}{\beta} \right) \right) \right\} \tag{2.2h}$$

$$\frac{d \log L}{d\alpha} = -\sum \left(-\frac{1}{\beta} + \frac{1}{\beta} \exp\left(-\frac{x_i - \alpha}{\beta} \right) \right) \tag{2.2i}$$

$$\frac{d \log L}{d\beta} = -\frac{n}{\beta} - \sum \left\{ -\frac{x_i - \alpha}{\beta^2} + \frac{x_i - \alpha}{\beta^2} \exp\left(-\frac{x_i - \alpha}{\beta} \right) \right\} \tag{2.2j}$$

$$= -\frac{n}{\beta} + \sum \frac{(x_i - \alpha)}{\beta^2} - \sum \frac{(x_i - \alpha)}{\beta^2} \exp\left(-\frac{x_i - \alpha}{\beta} \right) \tag{2.2k}$$

Choose α, β such that $\begin{bmatrix} dL/d\alpha \\ dL/\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using Newton-Raphson algorithm.

3.0 RESULTS AND DISCUSSIONS

Table 1. Median r values

n	Pareto	Gumbel Minimum
20	.594	.675
100	.678	.675
1000	.915	.675

Table 1 presents median r for ranks assigned 36 continuous distributions by the Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) methods. For the Pareto distribution, a coefficient of .594 was observed for n=20. The value increased to -678 for n=100 and to .915 for n=1000, indicating that increased sample size resulted in better agreement between the rankings of the continuous distributions by the two methods. For n=1000, the agreement is very strong. For Gumbel Minimum, the same agreement was exhibited for all sample sizes, indicating insensitivity to sample size variation. For all sample sizes, a fairly strong agreement of .675 was maintained.

Table 2. Presentation of best five distributions (n = 20 for Pareto)

Rank	K-S	A-D
1	Frechet (3P) α = 4.3905, $\beta = 0.90478$	Frechet (3P) α = 4.3905, $\beta = 0.90478$
2	log-logistic(3P) $\alpha = 21.162, \beta = 0.6067$	$\log normal(3p)\sigma = 0.46064, \mu = 0.45481$
3	Welbull (3P) $\alpha = 0.87077, \beta = 0.0277$	log-logistic(3P) $\alpha = 21.162, \beta = 0.6067$
4	$\log normal(3p)\sigma = 0.46064, \mu = 0.45481$	Inv.Gaussian (3P) $\lambda = 0.048, \mu = 0.629, \gamma = 0.578$
5	Gen.Extreme v K= .982, σ = .0108, μ = 586	Cauchy σ = 0.0143, μ = 0.5935

Table 2 presents the best five models as ranked by each of K-S and A-D for the Pareto case. Both methods picked Frechet as the best fit for r values except that K-S picked the 2-parameter Frechet while A-D picked the 3-parameter Frechet. K-S rated 3-parameter log-logistic as the second while A-D rated it as the third best. Both methods rated three distributions (3-parameter Frechet; 3-parameter log-logistic; 3-parameter log-normal) in common among their best five.

Table 3 presents the Pareto case for n=100. While K-S rated 2-parameter Frechet as the best, A-D rated it as the fourth. Both methods rated uniform distribution as the fifth best. K-S rated 3-parameter Frechet as the third best while A-D rated it as the second best. Both methods share four distributions in common among their best five.

Table 3. Presentation of best five distributions (n = 100 for Pareto)

Rank	K-S	A-D
1	Frechet (3P) α = 13.639, β = 68839	Beta α_1 = 0.12291, α2 = .13024 a = .667 b = .785
2	log-logistic(3P)α =1.249, β = .017 γ = .665	Frechet (3P) α = 1.073, β = .0175 γ = .665
3	Frechet (3P) α = 1.073, β = .0175 γ = .665	log-logistic(3P)α =1.249, β = .017 γ = .665
4	Inv.Gaussian λ = 133.8, μ = .7196	Frechet α = 13.639, β = 68839
5	Uniform a= .62823 , b = .81107	Uniform a= .62823 , b = .81107

Table 4 presents the case of n=1000. K-S and A-D seem to have improved agreement in their ranking. Both rated Generalized extreme and Gumbel Minimum as the best and second best. They share four distributions in common among their best five.

Table 4. Presentation of best five distributions (n = 1000 for Pareto)

Rank	K-S	A-D
1	Gen.Extreme v K= .581, σ = .048, μ = 065	Gen.Extreme v K= .581, σ = .048, μ = 065
2	Gumbel Min σ = .0368, μ = 0.9329	Gumbel Min σ = .0368, μ = 0.9329
3	Log-logistic (3P) α = 4.323, β = 1.0506 γ = 1.0506	Weibull α = 8.8272, β = 3.0627 γ = 3.0627
4	Weibull α = 8.8272, β = 3.0627 γ = 3.0627	Beta α1 = 0.12291, α2 = .13024 a = .667 b = .785
5	lognormal (3P) α.=0169 μ = .994 γ = 1.788	Log-logistic (3P) α = 4.323, β = 1.0506 γ = 1.0506

Table 5 presents the case of Gumbel Minimum for sample size 20. Both methods rated Log-logistic as the best. The two methods share four distributions in common. Table 6 the case of n=100. K-S and A-D respectively rated 3-

parameter Log-logistic and 3-parameter Frechet as the best. They share three distributions in common.

Table 5. Presentation of best five distributions (n = 20 for Gumbel Minimum)

Rank	K-S	A-D
1	Gen.Extreme v K= .581, σ = .048, μ = 065	Gen.Extreme v K= .581, σ = .048, μ = 065
2	Gumbel Min σ = .0368, μ = 0.9329	Gumbel Min σ = .0368, μ = 0.9329
3	Log-logistic (3P) α = 4.323, β = 1.0506 γ = 1.0506	Weibull α = 8.8272, β = 3.0627 γ = 3.0627
4	Weibull α = 8.8272, β = 3.0627 γ = 3.0627	Beta α1 = 0.12291, α2 = .13024 a = .667 b = .785
5	lognormal (3P) α.=0169 μ = .994 γ = 1.788	Log-logistic (3P) α = 4.323, β = 1.0506 γ = 1.0506

Table 6. Presentation of best five distributions (n = 100 for Gumbel Minimum)

Rank	K-S	A-D
1	Log-logistic α = 18.854, β = 0.69133	Log-logistic α = 18.854, β = 0.69133
2	Log-logistic (3P) α = 5.3073, β = 0.2562 γ = 0.4352	Frechet α = 12.627, μ = 0.66134
3	Cauchy σ = 0.02801, μ = 0.6613	Log-logistic (3P) α = 5.3073, β = 0.2562 γ = 0.4352
4	Frechet α = 12.627, μ = 0.66134	Cauchy σ = 0.02801, μ = 0.6613 a = .667 b = .785
5	Weibull α = 9.2285, β = 0.4684 γ = 0.2492	Weibull α = 9.2285, β = 0.4684 γ = 0.2492

Table 7 presents for n=1000. The Generalized extreme distribution rated second by K-S is rated best by A-D while Pert rated best by K-S is rated fourth by A-D. The methods shared four distribution s in common. In all but one case, four distributions are shared in common among the best five of distributions fitted to the values of r.

Table 7. Presentation of best five distributions (n = 1000 for Gumbel Minimum)

Rank	K-S	A-D
1	Pert m= 0.932 a= 0.7677 b = 0.9831	Gen.Extreme v k=0.5225 σ = 0.0405, μ = 0.9056
2	Gen.Extrem v k=0.522 σ = 0.040, μ = 0.90	Beta α1 = 0.12291, α2 = .13024 a = .667 b = .785
3	Beta α1 = 0.12291, α2 = .13024 a = .667 b = .785	Johnson SP γ = 0.2492 λ = 0.3165 μ = 0.9056
4	Johwnson SP γ = 0.2492 λ = 0.3165 μ = 0.9056	Pert α1 = 0.12291, α2 = .13024 a = .667 b = .785
5	Gen.Pareto k= 1.5722 σ = 0.193, μ = 0.839	Weibull α = 28.934, β = 0.4684 γ = 0.93078

Table 8 presents the results of Mann-Whitney test where the alternative hypothesis is that the median rank assigned by the K-S is less than that assigned by the A-D. Median ranks assigned Pareto by K-S and A-D are 10 and 26 respectively for n=20. The null hypothesis is rejected at α=0.01. Hence the median rank assigned by K-S is less than that assigned by A-D. K-S has hence performed better in identifying the Pareto. For n=100, the median ranks are respectively 6.5 and 11 for K-s and A-D. The test is rejected at α=0.01, indicating that K-S is better. For n=1000, the median ranks are 2 and 5 for K-S and A-D respectively. The test is also significant the null hypothesis is rejected at α=0.01. For both methods, the median rank assigned Pareto decreased with increase in sample size. This signifies improvement by both methods as sample size increased.

Table 8. Results of hypothesis testing for Pareto

n=20				n=100				n=1000			
Median		p-val	W	Median		p-val	W	Median		p-value	W
K-S	A-D			K-S	A-D			K-S	A-D		
10	26	0.000	6062	6.5	11	.000	6440	2	5	.000	7510

Table 9 presents the results for Gumbel Minimum where the alternative hypothesis is that the median rank assigned by K-S is greater than that assigned by A-D. The median ranks assigned the Gumbel by K-S and A-D are respectively 10 and 8 for n=20; 9 in each case when n=100 and 8 and 7 for n=1000. For n=20, the test is significant. The null hypothesis that the median rank for K-S is not greater than that of A-D is rejected at α=0.01. A-D is hence better at identifying Gumbel Minimum when n=20. For n=100, the test is also significant and can be rejected at α=0.05, still signifying better performance by A-D. For n=1000, the situation is not same as the test is not significant. This means that A-D is not better for n=1000.

Table 9. Results of hypothesis testing for Gumbel Minimum

n=20				n=100				n=1000			
Median		p-val	W	Median		p-val	W	Median		p-value	W
K-S	A-D			K-S	A-D			K-S	A-D		
10	8	.0001	11594	9	9	.010	101999	8	7	.01889	10411.

Table 10. Frequencies of correctly identified cases

Pareto						Gumbel Minimum					
n=20		n=100		n=100		n=20		n=100		n=1000	
K-S	A-D	K-S	A-D	K-S	A-D	K-S	A-D	K-S	A-D	K-S	A-D
2	0	9	0	9	3	2	0	0	2	4	4

Table 10 presents the number of times (out of 100 replications) the distribution (Pareto or Gumbel Minimum) is correctly identified (rated first) by each of K-S and A-D. K-S identified Pareto correctly on 2, 9 and 9 occasions for sample size 20, 100 and 1000 respectively. For n=20 and n=100, A-D could not identify the Pareto correctly while it identified it correctly on 3 occasions for n=1000. Hence, K-S has identified Pareto greater number of times than A-D- (a pointer to supremacy in identifying Pareto). For Gumbel Minimum case, K-S identified the simulated distribution as Gumbel Minimum 2 times for n=20; on no occasion for n=100 and 4 times for n=1000. On Gumbel, the two methods exhibited equal performance.

4.0 CONCLUSION

The study has compared the performance of Kolmogov-Smirnov and Anderson-Darling goodness of fit tests on Pareto and Gumbel Minimum densities. Results revealed: supremacy of Komogorov-Smirnov test over Anderson-Darling test for Pareto (for all sample sizes) and supremacy of Anderson-Darling test over Kolmogorov-Smirnov test for Gumbel Minimum (for small and moderate sample sizes); stalemate between K-S and A-D for large n (n=1000) in the case of Gumbel Minimum; improved agreement between K-S and A-D with increasing sample size for case of Pareto; constant agreement between K-S and A-D for all sample sizes in the case of Gumbel Minimum. It is recommended that more distributions and more goodness of fit tests be investigated.

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