ON GEODESIC LINES OF METRIC SEMI-SYMMETRIC CONNECTION OF HYPERBOLIC ALMOST KAEHLERIAN SPACES

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ABSTRACT

Goldberg (1956) has studied on projectively Euclidean Hermitian spaces. Mizusawa and Koto (1960) have studied holomorphically projective curvature tensors in certain almost Kaehlerian spaces. Also, Prvanovic and Pusic (1995) have studied on manifold admitting some semi-symmetric metric connection. In the present paper, we have defined and studied Geodesic lines on any metric space are autoparallel lines of its Levi-Civita connection. The necessary and sufficient condition for a metric semi-symmetric connection of a hyperbolic almost Kaehlerian space to have some of their autoparallel lines in common with their Levi-Civita connection.

KEY WORDS AND PHRASES: Hyperbolic Kaehlerian spaces, Geodesic lines, metric semi-symmetric connection, gradient, generating and isotropic vector field.
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1. INTRODUCTION.

Let us consider a Riemannian space M, of dimension **n**, with metric tensor (g_{ij}). Let us denote the Christoffel symbols towards given metrics by $\{^{\,i}_{j\,k}\}$, the operator of covariant differentiation towards the Levi-Civita connection by $\dot{\nabla}$ and components of its curvature tensor by K^i_{jkl} (or by K_{ijkl} for its Riemann-Christoffel tensor).

The geodesic line, that means, an autoparallel line of Levi-Civita connection is characterized by the relation

(1.1) $v^k \dot{\nabla}_k v^j = 0,$

Where v^k stands for a component of tangent vector field of the geodesic line.

The component of metric semi-symmetric connection are given by

(1.2) $\Gamma^a_{ik} = \{^{\,a}_{i\,k}\} + p_i \delta^a_k - p^a\ g_{ik}$

Where p_i *and* p^a are covariant and contravariant components of a vectors of a vector field. This vector field is called the generator or the generating vector field of metric semi-symmetric connection. The torsion tensor components of the metric semi-symmetric connection are equal

(1.3) $$T^{a}_{ik} = p_i \delta^a_k - p^k \delta^a_i$$

Let v^k be the component of tangent vector field of a geodesic line, then, this geodesic line will be autoparallel for metric semi-symmetric connection if and only if three holds

(1.4) $$p_k \, v^k v^j \;=\; p^j v_k \, v^k$$

As we consider a Riemannian space and its metrics is positively definite, (1.3) reduces if and only if

(1.5) $$p_k \;=\; \alpha \, p_k$$

Where α is a scalar function.

If we explore properties of the function α, as it is the proportionality coefficient between the tangent vector field of the geodesic line and generating vector field of the metric semi-symmetric connection, we shall get an answer to the following question; for a geodesic line on a Riemannian space, how many semi-symmetric metric connections have same line as an autoparallel line.

Let us denote the operator of covariant differentiation towards metric semi- symmetric connection by ∇, then

$$\nabla_k v_j \;=\; \dot{\nabla}_k v_j - \alpha\left(v_k v_j - g_{kj}\right) v$$
$$\dot{\nabla}_k p_j \;=\; \alpha_k v_j \;+ \alpha \, \dot{\nabla}_k v_j ,$$

Where v stands for scalar square of the vector v_k and $\alpha_k = {\partial \alpha}/{\partial x^k}$.

The components of Riemannian-Christoffel tensor of the metric semi-symmetric connection can be expressed in this way

(1.6) $$R_{ijkl} = K_{ijkl} + g_{ik} p_{lj} - g_{il} p_{kj} + g_{jl} p_{ki} - g_{jk} p_{li}$$

Where K_{ijkl} denotes a Riemann-Christoffel tensor component of Levi-Civita connection and the abbreviation p_{kj} stands for the tensor

(1.7) $$p_{kj} = \dot{\nabla}_k p_j - p_j p_k + \frac{1}{2} \, p_s \, p^s g_{jk} .$$

The tensor p_{kj} is symmetric if and only if p_j is a gradient, that means

$$\alpha_k v_j - \alpha_j v_k = \alpha(\dot{\nabla}_j v_k - \dot{\nabla}_k v_j),$$

Or, equivalently

(1.8) $\frac{\partial v}{\partial x^k} = \frac{2}{\alpha} \, (\mu v_k - v\alpha_k)$

Where μ stands for $\alpha_k v^k$.

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From the expression (1.5), we easily obtain,

$$R_{jk} = K_{jk} + (2-n)p_{kj} - g_{jk}\, p_s^s,$$
$$g_{kj}\, p_s^s = K_{jk} - R_{jk} - (n-2)p_{jk},$$
$$np_s^s = K - R - (n-2)\, p_s^s,$$
$$p_s^s = {(K-R)}\Big/{2(n-1)},$$
$$p_{kj} = \frac{K_{jk} - R_{jk}}{n-2} - \frac{K-R}{2(n-1)(n-2)} g_{jk}\,.$$

By K_{jk}, K *and* R_{jk}, R we denote the Ricci tensor and the curvature scalar for the Levi-Civita connection and for metric semi-symmetric connection respectively.

We have our curvature tensor (1.5) to satisfy the all algebraic properties which are most common for curvature tensors to be skew-symmetric in first two Indies, to be invariant under change of places of first and second pair of indices and to satisfy the first Bianchi identity. All these properties are satisfied if and only if the generating vector field is a gradient.

There holds

$v^i \dot{\nabla}_i v_k = 0$ and $p^i \dot{\nabla}_i p_k = p^i \dot{\nabla}_k p_i = \varphi p_k = \varphi \alpha v_k.$

Now, we apply the Ricci identity for the metric semi-symmetric connection to the generator and we obtain

(1.9) $v\alpha_j - \varphi v_j = 0.$

Then there yields, in view of (1.7), $\frac{\partial v}{\partial x^k} = 0,$

and the tangent vector of the geodesic line is of constant length.

From (1.8), we have $\alpha_k = \frac{\varphi}{v} v_k,$ or $\alpha_k = f\, p_k.$

This means that all three vectors are mutually proportional. Then

$$p_s p^s = \alpha^2 v.$$

Besides

$$\frac{\partial (p_s p^s)}{\partial x^k} = p^s \dot{\nabla}_k p_s + p_s \dot{\nabla}_k p^s = p^s \alpha_k p_s + \alpha p^s \dot{\nabla}_k v_s$$
$$= 2\alpha_k p_s p^s + 2\alpha p^s \dot{\nabla}_k v_s = 2\alpha_k p_s p^s + 2\alpha^2 v^s \dot{\nabla}_k v_s$$
$$= 2 p_s p^s \alpha_k = 2\alpha^2 \alpha_k v.$$

As $\dot{\nabla}_k v^s v_s = 0$ and consequently $v^s \dot{\nabla}_k v_s = 0,$

Then, on the other side

$$\frac{\partial(p_s p^s)}{\partial x^k} = 2p^s \dot{\nabla}_k p_s = 2\alpha_k p^s p_s = 2\alpha^2 v \alpha_k$$
$$= 2\alpha v\, \alpha_k.$$

Comparing two results for $\frac{\partial(p_s p^s)}{\partial x^k}$, we obtain $\alpha = 1$ or $\alpha_k = 0$. Then
(1) (p_k) and (v_k) are equal, both gradients, both of constant length,
Or
(2) (p_k) and (v_k) are collinear vectors, both of constant length and both gradients. Therefore

Definition (1.1): On a Riemannian space, the curvature tensor of metric semi-symmetric metric connection satisfies the all most common algebraic properties for any curvature tensor if and only if the generating vector field is a gradient. Then the Levi-Civita connection and metric semi- symmetric connection have some of their autoparallel lines in common if the generating vector field of metric semi-symmetric connection and tangent vector field of the geodesic line are collinear gradients of constant length.

2. HYPERBOLIC ALMOST KAEHLERIAN SPACE

A hyperbolic almost Kaehlerian space is an even-dimensional pseudo-Riemannian space, endowed with a nondegenerate structure tensor F_j^i satisfying

(2.1) $$F_j^i F_k^j = \delta_k^i, \qquad F_{ij} = -F_{ji}, \qquad \dot{\nabla} F_{ij} = 0.$$

Theorem (2.1): The curvature scalars of Levi-Civita connection, the Riemannian part of metric semi-symmetric connection and metric semi-symmetric F-connection are mutually equal.

Proof: Consider a semi-symmetric on a hyperbolic almost Kaehlerian space has the torsion

(2.2) $$T_{\boldsymbol{ij}}^k = p_i \delta_{\boldsymbol{j}}^{\boldsymbol{k}} - p_{\boldsymbol{j}} \delta_{\boldsymbol{i}}^{\boldsymbol{k}} + q_{\boldsymbol{i}} K_{\boldsymbol{j}}^{\boldsymbol{k}} - q_{\boldsymbol{j}} F_{\boldsymbol{i}}^{\boldsymbol{k}},$$

where p_i and q_i are components of certain vector fields. If we, moreover, want this connection to be a metric one, then it has components

(2.3) $$H_{ik}^a = \left\{ {a \atop i \;\; k} \right\} + p_i \delta_k^a - p^a g_{ik} - q_k F_i^a$$

The connection ∇ is an F-connection, that means that $\nabla F = 0$. Then

(2.4) $$q_j = -\frac{n}{2} p_a F_j^a \;, \quad p_a F_j^a = -\frac{2}{n} q_j$$

Then we can denote

(2.5) $$H_{ik}^a = \Gamma_{ik}^a - q_k F_i^a$$

where Γ_{ik}^{a} is a component of Riemannian part of metric semi-symmetric F-connection, which is itself a component of a metric semi-symmetric connection on the adjoint pseudo-Riemannian space, satisfying conditions of the definition(1.1).

Now we can calculate the coefficients of curvature tensor of connection (2.3)

$$\bar{R}_{ijkl} = R_{ijkl} - F_{ji}\left(\dot{\nabla}_l q_k - \dot{\nabla}_k q_l\right) + \\ + q_k\left(p_j F_{li} + \frac{2}{n} q_j g_{li} + \frac{2}{n} q_i g_{lj} + p_i F_{jl}\right) - \\ - q_l\left(p_j F_{ki} + \frac{2}{n} q_j g_{ki} + \frac{2}{n} q_i g_{kj}\right) + p_i F_{jk}$$

By R_{ijkl} we denote a component of curvature tensor of metric semi-symmetric connection, satisfying of definition (1.1) and the tensor $\bar{R}_{ijkl}$ is skew-symmetric in first two Indies. $\bar{R}_{ijkl}$ is invariant under changing places of first and second pair of indies if and only if the tensor ($p_l q_k + q_l p_k$) is skew-symmetric. Then

$$p_k p^k q_l = -p^k q_k p_l .$$

As the vectors p^k and q^k are mutually orthogonal, there yields $p_k p^k = 0$. This means that the generator of metric semi-symmetric connection, that is, the Riemannian part of metric semi-symmetric F-connection is an isotropic gradient, which is in accordance with the statement of definition (1.1). Then the vector q_k is also an isotropic vector.

Theorem (2.2): The curvature tensor of a metric semi-symmetric F-connection on the hyperbolic almost Kaehlerian space is invariant under changing places of first and second pair of indices and satisfies the first Bianchi identity if and only if the generators of the connection are isotropic and $\dot{\nabla}_a p^a = 0$. Then the all geodesic lines whose tangent vectors are proportional to the generators or Eigen for the structure are autoparallel for the metric semi-symmetric F-connection, and conversely.

Proof: We have the autoparallel lines of the connection (2.3) are geodesic lines of the adjoint pseudo-Riemannian space if and only if the condition

(2.6) $$p_j \delta_k^i v^j v^k - p^i q_j v^j v^k - q_k F_j^i v^j v^k = \frac{2}{n} p_j v^j v^i$$

where v^i is the tangent vector field of a geodesic line. Then

(2.7) $$(p_j v^j - v_j v^j - \frac{2}{n} p_j v^j)\, v^i = -q_k v^k u^i,$$

where $$u^i = -F^{ji} v_j = F^{ij} v_j .$$

Then, from (2.7),

(2.8) $u^i = \alpha\, v^i,$ or $q_k v^k = 0.$

If (2.7) holds, then the tangent vector field is eigen for the structure, for one of its eigen values, **1 or -1.** Then v_k is a self-orthogonal or isotropic vector field. The scalar product $q_k v^k$ then equals to

$$\frac{n-2}{n} p_j v^j - v_j v^j = \frac{n-2}{n} p_j v^j .$$

If (2.8) holds, we have express the vectors in the adapted basis

(2.9) $$p = p^a l_a + p^{\hat{b}} l_{\hat{b}} ,$$

where $l_{\hat{b}}$ are also eigen vector, for the eigen values **-1.** Then

$$q = -\frac{2}{n} p^a l_a + \frac{2}{n} p^{\hat{b}} l_{\hat{b}} \quad \text{and} \quad v = v^a l_a + v^{\hat{b}} l_{\hat{b}}.$$

Then (2.8) gives

(2.10) $$q_k v^k = \frac{2}{n}\left(p^{\hat{b}} v^a - p^a v^{\hat{b}}\right) g_{a\hat{b}} = 0,$$

It is satisfied if and only if v is proportional to p. Anyway, the tangent vector field of the geodesic-autoparallel line is isotropic.

As for the hyperbolic almost Kaehlerian space the generating vector field of the metric semi-symmetric *F*-connection having some of its autoparallel lines in common with Levi-Civita connection is isotropic, then the tensor (1.6) looks this way

(2.11) $$p_{kj} = \dot{\nabla}_k p_j - p_k p_j$$

and

(2.12) $$\dot{\nabla}_s p^s = p_s^s = \frac{K-R}{2(n-1)} .$$

Contracting the tensor ($\dot{\nabla}_k q_l - \dot{\nabla}_l q_k$) with the tensor F_b^l, we obtain

$$-\frac{2}{n}\dot{\nabla}_k p_b + \frac{2}{n} F_b^l F_k^a \dot{\nabla}_l p_a = -\frac{n}{2}\dot{\nabla}_k p_b + F_b^l \dot{\nabla}_l q_k$$

and $$\frac{n-4}{2n}\dot{\nabla}_k p_b = F_b^l\left(\dot{\nabla}_l q_k - \frac{2}{n} F_k^a \dot{\nabla}_l p_a\right).$$

Contracting the last relation with g^{kb}, we obtain

$$\frac{n-4}{2n}\dot{\nabla}_a p^a = -\frac{2}{n}\dot{\nabla}_a p^a + \frac{2}{n} p_a p^a = 0.$$

If $n > 4$, then

(2.13) $\dot{\nabla}_a p^a = 0.$ Then, by (2.12), $K = R$.

However, from (2.5) and the form of curvature tensor, we can obtain that

(2.14) $\bar{R} = R + F^{lk} \left(\dot{\nabla}_l \, q_k - \dot{\nabla}_k q_l \right)$

or $\bar{R} = R + \frac{2}{n} \left(\dot{\nabla}_l \, p^l + \dot{\nabla}_l \, p^l \right) = R + \frac{4}{n} \dot{\nabla}_l \, p^l.$

If we have our curvature tensor to satisfy the first Bianchi identity, using expression for the curvature tensor, then we also obtain and proved $\boldsymbol{\dot{\nabla}_a \, p^a = 0.}$

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