HASH SEMI-OPEN SETS

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ABSTRACT

An ideal space is a triplet (X, τ, I), where X is a non empty set, τ is a topology on X and I is an ideal of subsets of X. In this paper, we introduce and study Hash semi-open sets in an ideal space.

Key words*: *Semi-open set, Hash semi open set.

1. INTRODUCTION

The contributions of Hamlett and Jankovic [5, 6, 7, 8] in ideal topological spaces initiated the generalization of some important properties in General Topology via Topological ideals. The properties like decomposition of continuity, covering property, separation axioms, connectedness, extremal disconnectedness, compactness and resolvability [1, 2, 3, 4, 12] have been generalized using the concepts of ideals in topological spaces. Manoharan and Thangavelu [11] applied the concept of ideals in the field of Graph theory to characterize Eulerian graphs.

By a space (X,τ), we mean a topological space X with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X,τ), the system of open neighborhoods of x is denoted by $N(x) = \{ U \in \tau : x \in U\}$.

For a given subset A of a space (X, τ), *cl* (A) and *int* (A) are used to denote the closure of A and interior of A respectively with respect to the topology τ.

A non empty collection of subsets of a set X is said to be an ideal on X, if it satisfies the following two conditions: (i) If $A \in I$ and $B \subseteq A$, then $B \in I$; (ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal space (X, τ, I) means a topological space (X, τ) with an ideal I defined on X. Let (X, τ) be a topological space with an ideal I defined on X. Then for any subset A of X, $A^*(I, \tau) = \{x \in X / A \cap U \notin I$ for every $U \in N(x)\}$ is called the local function of A with respect to I and τ [9]. If there is no ambiguity, we will write A*(I) or simply A* for A*(I, τ). Also, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for the topology τ*(I) (or simply τ*) which is finer than τ.

We use the following ideals in our discussion.

(i) I = $\{\emptyset\}$, the ideal containing empty set only.

(ii) $\wp(X)$ = the power set of X = the ideal of all subsets of X.

(iii) I_F = the ideal of finite sub sets of X.

(iv) Ic = the ideal of countable sub sets of X.

2. HASH SEMI-OPEN SETS

A subset A of a space (X, τ) is semi-open (semi-closed)[10] if $A \subseteq cl(\text{int}(A))$ ($A \supseteq cl(int(A))$). Semi-closure of the subset A is the intersection of all semi-closed sets containing A and it is denoted by *scl* (A). We denote the class of all semi-open sets in (X, τ) containing $x \in X$ by SO(X, τ, x). In this section, we introduce and study a new local function $A^{\#}$ for all the subsets A of an ideal topological space (X, τ, I).

Definition 2.1

Let (X, τ) be a topological space with an ideal I on X and $A \subseteq X$. Then, $A^{\#}(I, \tau) = \{ x: A \cap U \notin I$ for every $U \in S(X, \tau, x) \}$.

Since $N(x) \subseteq SO(X, \tau, x)$ }, it follows that $A^{\#}(I, \tau) \subseteq A^{*}(I, \tau)$.

Definition 2.2

Let (X,τ) be a topological space with an ideal I on X. For any subset A of X, $scl^{\#}(A) = A \cup A^{\#}(I, \tau)$.

Remark 2.3

In an ideal space (X, τ, I) ,

(i) $scl^{\#}(A) \subseteq cl^{*}(A)$.

(ii) If $A \in I$, then $A^{\#} = \phi = A^{*}$.

(ii) If $A \in I$, then $scl^{\#}(A) = A = cl^{*}(A)$.

Proposition 2.4

Let (X,τ) be a space with ideals I, I_1 and I_2 on X, and let A and B be any two subsets of X. Then

(i) $A \subseteq B \Rightarrow A^{\#} \subseteq B^{\#}$;

(ii) $I_1 \subseteq I_2 \Rightarrow A^{\#}(I_2) \subseteq A^{\#}(I_1)$;

(iii) $A^{\#} = scl(A^{\#}) \subseteq scl(A)$;

(iv) $(A^{\#})^{\#} \subseteq A^{\#}$;

(v) $(A \cup B)^{\#} = A^{\#} \cup B^{\#}$;

(vi) $(A \setminus B)^{\#} \setminus B^{\#} \subseteq A^{\#} \setminus B^{\#}$;

(vii) For every $B \in I$, $(A \cup B)^{\#} = A^{\#}$.

Proof

Suppose $A \subseteq B$. Let $x \in A^{\#}$. Then $U \cap A \notin I$ for every $U \in SO(X, \tau, x)$. Since $U \cap A \subseteq U \cap B$ and since I is an ideal, it follows that $U \cap B \notin I$ for every $U \in SO(X, \tau, x)$. Therefore, $x \in B^{\#}$ that proves (i).
Now, suppose $I_1 \subseteq I_2$. Let $x \in A^{\#}(I_2)$ that implies $U \cap A \notin I_2$ for every $U \in SO(X, \tau, x)$. This shows that $U \cap A \notin I_1$ for every $U \in S(X, \tau, x)$, completing the proof for (ii).
Since $scl^{\#}(A) = A \cup A^{\#}(I, \tau)$, $A^{\#} \subseteq scl(A^{\#})$. Let $x \in scl(A^{\#})$ that implies $U \cap A^{\#} \neq \phi$ for every $U \in SO(X, \tau, x)$. Fix $U \in SO(X, \tau, x)$. Let $y \in U \cap A^{\#}$. Since $y \in U$ and $y \in A^{\#}$, $U \cap A \notin I$ that implies $x \in A^{\#}$. Therefore $A^{\#} = scl(A^{\#})$. Since $U \cap A \notin I \Rightarrow U \cap A \neq \phi$, it follows that $scl(A^{\#}) \subseteq scl(A)$; This proves (iii).
Let $x \in (A^{\#})^{\#}$. Then $U \cap A^{\#} \notin I$ for every $U \in SO(X, \tau, x)$. In particular $U \cap A^{\#} \neq \phi$ for every $U \in SO(X, \tau, x)$. Fix $U \in SO(X, \tau, x)$ and $y \in U \cap A^{\#}$ that implies $U \cap A \notin I$. This proves that $x \in A^{\#}$ that completes the proof for (iv).
From (i), it follows that $A^{\#} \cup B^{\#} \subseteq (A \cup B)^{\#}$. To prove the other inclusion let $x \in (A \cup B)^{\#}$. Then $U \cap (A \cup B) \notin I$ for every $U \in SO(X, \tau, x)$. If $U \cap A \in I$ and
$V \cap B \in I$ for some $U, V \in SO(X, \tau, x)$ then
$(U \cap V) \cap (A \cup B) = (U \cap V \cap A) \cup (U \cap V \cap B) \in I$ that contradicts the choice of x. Therefore $U \cap A \notin I$ or $U \cap B \notin I$ for every $U \in SO(X, \tau, x)$. This shows that
$x \in A^{\#} \cup B^{\#}$ completing the proof for (v). (vi) is obviously true from (i).
From Remark 2.3(ii), $B^{\#} = \phi$ for every $B \in I$ that implies $(A \cup B)^{\#} = A^{\#} \cup B^{\#} = A^{\#} \cup \phi = A^{\#}$ for every $B \in I$. This proves (vii) and the proof for the proposition is completed.

Corollary 2.5

Let B be a subset of an ideal space (X, τ, I). Then ,$(X \setminus B)^{\#} \setminus B^{\#} \subseteq X^{\#} \setminus B^{\#}$.

Proof

Replacing A by X in Theorem 2.4(vi), we get this corollary.

Definition 2.6

Let A be a subset of an ideal space (X, τ, I). Then, A is $\tau^{\#}$-semi-closed if $scl^{\#}(A) = A$ and A is $\tau^{\#}$-semi-open if and only if $X \setminus A$ is $\tau^{\#}$-semi-closed. We will call a $\tau^{\#}$-semi-open set as Hash semi-open set.

It is clear that A is $\tau^{\#}$-semi-closed if and only if $A^{\#} \subseteq A$ and A is $\tau^{\#}$-semi-open if and only if $(X \setminus A)^{\#} \subseteq X \setminus A$ if and only if $X \setminus (X \setminus A)^{\#} \supseteq A$ if and only if $A \cap (X \setminus A)^{\#} = \phi$.

$SC^{\#}(X, \tau, I)$ denotes the collection of all $\tau^{\#}$-semi-closed sets in (X, τ, I) and $SO^{\#}(X, \tau, I)$ denotes the collection of all $\tau^{\#}$-semi-open sets in (X, τ, I).

Proposition 2.7

In an ideal space (X, τ, I),

(i) $SC^{\#}(X, \tau, I)$ contains X and ϕ, and is closed under arbitrary intersection.

(ii) $SO^{\#}(X, \tau, I)$ contains X and ϕ, and is closed under arbitrary union.

(iii) Every τ^{*}-closed set is $\tau^{\#}$-semi-closed.

(iv) Every τ^{*}-open set is $\tau^{\#}$-semi-open.

Proof

Obviously X and ϕ are $\tau^{\#}$-semi-closed sets. Suppose ∇ is a family of $\tau^{\#}$-semi-closed sets. Then for every $A\in\nabla$, $A^{\#} \subseteq A$. Then $\left(\bigcap_{A\in\nabla} A\right)^{\#} \subseteq \bigcap_{A\in\nabla} A^{\#} \subseteq \bigcap_{A\in\nabla} A$. This shows that $\bigcap_{A\in\nabla} A$ is $\tau^{\#}$-semi-closed. This proves (i). (ii) follows from (i) and Definition 2.6. (iii) and (iv) follows from the fact that $A^{\#} \subseteq A^{*}$.

Remark 2.8

If $I = \{\phi\}$, $SO^{\#}(X, \tau, I) = SO(X, \tau) = SO(X, \tau^*)$.

Remark 2.9

If $I = \wp(X)$, then $SO^{\#}(X, \tau, I) = \wp = SC^{\#}(X, \tau, I)$.

Proposition 2.10

If $I_1 \subseteq I_2$ then $SO^{\#}(X, \tau, I_1) \subseteq SO^{\#}(X, \tau, I_2)$

Proof

Since $I_1 \subseteq I_2$, by using Proposition 2.4, we get $A^{\#}(I_2) \subseteq A^{\#}(I_1)$.Now $A \in SC^{\#}(X, \tau, I_1) \Rightarrow A^{\#}(I_1) \subseteq A \Rightarrow A^{\#}(I_2) \subseteq A \Rightarrow A \in SC^{\#}(X, \tau, I_2)$.

This proves that $SO^{\#}(X, \tau, I_1) \subseteq SO^{\#}(X, \tau, I_2)$.

The following corollary is a direct application of the above theorem.

Corollary 2.11

In an ideal space (X, τ, I), $SO(X, \tau) \subseteq SO^{\#}(X, \tau, I) \subseteq \wp(X)$.

Notations

1. A^{sd} denotes the set of all semi-accumulation points of A. That is $x \in A^{sd}$ if and only if $(U\setminus\{x\})\cap A\neq\phi$ for every $U \in SO(X, \tau, x)$.

2. $A^{\omega s}$ denotes the set of all ω- semi-accumulation points of A. That is $x \in A^{\omega s}$ if and only if $U \cap A$ is infinite for every $U \in SO(X, \tau, x)$.

3. A^{sc} denotes the set of all semi-condensation points of A. That is $x \in A^{sc}$ if and only if $U \cap A$ is uncountable for every $U \in SO(X, \tau, x)$.

Proposition 2.12

Let A be a subset of an ideal space (X, τ, I) . Then

(i) $A^{sd\#} \subseteq A^{\#}$.

(ii) $A^{sd\#} \subseteq A^{sd}$.

(iii) If $\{x\} \in I$ then $x \in A^{sd\#}$ if and only if $x \in A^{\#}$.

(iv) $A^{sd\#} = A^{\#}(I_F)$.

Proof

$x \in A^{sd\#}$ if and only if $x \in scl^{\#}(A \setminus \{x\})$ if and only if $x \in (A \setminus \{x\}) \cup (A \setminus \{x\})^{\#}$ if and only if $x \in (A \setminus \{x\})^{\#}$ if and only if $(A \setminus \{x\}) \cap U \notin I$ for every $U \in SO(X, \tau, x)$. This proves that $A^{sd\#} \subseteq A^{\#}$.We also see that $x \in A^{sd\#}$ implies $(A \setminus \{x\}) \cap U \neq \phi$ for every $U \in SO(X, \tau, x)$ that shows that $A^{sd\#} \subseteq A^{sd}$. Suppose $\{x\} \in I$.

Then $x \in A^{\#} \Rightarrow A \cap U \notin I$ for every $U \in SO(X, \tau, x)$.

$\Rightarrow A \cap U \neq \phi$ and $A \cap U \neq \{x\}$ for every $U \in SO(X, \tau, x)$.

$\Rightarrow (A \setminus \{x\}) \cap U \neq \phi$ for every $U \in SO(X, \tau, x)$.

$\Rightarrow x \in A^{sd\#}$

This proves (iii) and (iv).

Proposition 2.13

Let A be a subset of an ideal space (X, τ, I) . Then

(i) $A^{\omega s} = A^{\#}(I_F)$

(ii) $A^{cs} = A^{\#}(I_C)$

Proof

$x \in A^{\omega s}$ if and only if $U \cap A$ is infinite for every $U \in SO(X, \tau, x)$ if and only if $A \cap U \notin I_F$ for every $U \in SO(X, \tau, x)$ if and only if $x \in A^{\#}(I_F)$. This proves (i).

$x \in A^{cs}$ if and only if $U \cap A$ is uncountable for every $U \in SO(X, \tau, x)$ if and only if $A \cap U \notin I_C$, for every $U \in SO(X, \tau, x)$ if and only if $x \in A^{\#}(I_F)$. This proves (ii).

Proposition 2.14

Let I_F be the ideal of all finite subsets of (X, τ). Then $A^{\omega s} = A^{\#}(I_F) = A^{sd\#}$.

Proof:

Now $\{x\} \in A^{\#}(I_F)$ if and only if $U \cap A$ is infinite for every $U \in SO(X, \tau, x)$ if and only if $U \cap A \notin I_F$ for every $U \in SO(X,\tau,x)$. Therefore, $A^{\omega s} = A^{\#}(I_F)$. By Proposition 2.14 (iv), $A^{\#}(I_F) = A^{sd\#}$. Thus, $A^{\omega s} = A^{\#}(I_F) = A^{sd\#}$.

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