Chebyshev Expansion Method for the Solution of Polynomial and Non-Polynomial Variable Coefficients Differential Equations

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Abstract
This paper presents Chebyshev expansion method for the solution of polynomial and non-polynomial variable differential equations. In such problems, where the function is non-polynomial, then Chebyshev polynomial is needed to express the product in series for easy computation of sides of the differential equation. Numerical examples are carried on notable functions and the results are presented.

Key words: Chebyshev Expansion, Differential Equation, Non-Polynomial, Taylor Series

1.0 Introduction
The computation of a function \( f(x) \) for one or more given values of its argument \( x \) is one of the most common problems in numerical analysis. The function may as well occur in its products with other functions say \( G(x) \) and in such cases we are confronted with the challenge of expressing these products in computable form. Also functions may be defined in a variety of ways, either explicitly, as a series, a definite integral or some other computable form; or implicitly for example as the solution of a differential or integral equation [2], [4].

According to [3], Chebyshev expansion method fails because of the inability to express variable coefficient problems in its Chebyshev series equivalence before the process of coefficients comparison. For polynomial variable coefficients differential equations, a standard technique is given in [3] where the use of the Chebyshev recurrence relation \( G_{m+1}(x) = 2xG_m(x) - G_{m-1}(x) \) for expressing such products in Chebyshev form is thoroughly explained.

[1] used collocation method for differential equation that centered on non-polynomial coefficients problems and those that lead to polynomial variable coefficient s problems adopted the use of Chebyshev expansion method. [6] considered Chebyshev series representation for product of Chebyshev polynomial and some notable functions.

In this paper, we shall make use of Chebyshev method expansion to approximate the solution of polynomial and non-polynomial variable coefficient differential equations.

2.0 On Chebyshev Polynomials
The literature on Chebyshev polynomials is enormous. Here we shall supply the barest essentials in an effort to keep this paper self-contained. We presented the definition of Chebyshev polynomials and some basic properties needed in illustrating our main results.
A standard definition of the Chebyshev polynomial of the first kind denoted by $G_m(x)$ and are defined as a set of orthogonal polynomial of degree $m$ given by

$$G_m(x) = \cos \left( m \cos^{-1} \left( \frac{2x-b-a}{b-a} \right) \right)$$

and satisfies the recurrence relation;

$$G_{m+1}(x) = 2 \left( \frac{2x-b-a}{b-a} \right) G_m(x) - G_{m-1}(x)$$

which is valid in the interval $[a, b]$.

We also define Chebyshev polynomials of the second kind denoted by $H_m(x)$, which is defined in a number of ways, one of which is:

$$H_m(x) = \frac{1}{m+1} G''_{m+1}(x)$$

And satisfies the recurrence relation of the form

$$H_{m+1}(x) = 2xH_m(x) - H_{m-1}(x)$$

### 2.1 Properties of the Chebyshev polynomials $G_m(x)$

Chebyshev polynomial is enormous especially in the field of numerical approximation for interval $[-1, 1]$ which satisfies the following properties below [6]

i. $|G_m(x)| \leq 1$

   From equation (5), $G_m(x)$ has $m$ zeros lie in the interval $[-1, 1]$, they are given by

   $$x_j = \frac{\cos(k\pi)}{m}, j = 0, 1, 2, ..., m$$

   At these points, the values of the polynomials are

   $$G_m(x_j) = (-1)^j$$

ii. The leading coefficient of $x^m$ in $G_m(x)$ is $2^{m-1}$ and $G_m(-x) = (-1)^m 2^{m-1} = (-1)^m G_m(x)$.

iii. The maxima and minima are spread reasonably uniformly.

iv. Chebyshev polynomials are easy to compute and to convert to power series vice versa.

v. They satisfy a three term recurrence relation of the form

   $$G_{m+1}(x) = 2xG_m(x) - G_{m-1}(x), n = 1, 2, 3, ... ,$$

   with starting values $G_0(x) = 1, G_1(x) = x$

   Explicit expressions for the Six Chebyshev polynomials are;

   $$G_0(x) = 1, \ G_1(x) = x,$$

   $$G_2(x) = 2x^2 - 1, \ G_3(x) = 4x^3 - 3x,$$

   $$G_4(x) = 8x^4 - 8x^2 + 1, \ G_5(x) = 16x^5 - 20x^3 + 5x$$
The above properties produce an approximate polynomial which brings about error reduction in its application, this is different from the least squares approximation where the sum of the squares of the errors is minimized; the maximum itself can be quite large. But in Chebyshev approximations, the average error can be large but the maximum error is minimized [6].

We shall use the above properties to prove the theorem below:

**Theorem 1 [6]**

The polynomial \( \tilde{G}_m(x) = 2^{1-m}G_m(x) \) is the minimax approximation on \([-1,1]\) to the zero function by a monic polynomial of degree \( m \) and \( \|G_m(x)\| \leq 2^{1-m} \)

**Proof:** let us suppose that there exists a monic polynomial \( P_m \) of degree \( m \) such that

\[
|P_m(x)| \leq 2^{1-m}
\]

for all \( x \in [-1,1] \) and we arrive at a contradiction.

Let \( x_j, j = 0,1,\ldots,m \) be the abscissa of the extreme of the Chebyshev polynomial of degree \( m \).

Because of property (ii), then

\[
P_m(x_0) < 2^{1-m}G_m(x_0),
\]

\[
P_m(x_i) < 2^{1-m}G_m(x_i),
\]

\[
P_m(x_j) < 2^{1-m}G_m(x_j),
\]

Therefore, the polynomial \( Q(x) = P_m(x) - 2^{1-m}G_m(x) \) changes sign between each two consecutive extreme of \( G_m(x) \). Thus it changes sign \( m \) times. But it is not possible because \( Q(x) \)
is a polynomial whose degree is smaller than \( m \). (It is a subtraction of two monic polynomials of degree \( m \)).

### 2.2 Numerical Procedures and Implementation for Chebyshev approximations

We shall consider a linear problem with linear associated conditions, whose coefficients and terms are themselves polynomials given below

\[
xG_m(x) = \frac{1}{2} \left( G_{m+1}(x) + G_{m-1}(x) \right)
\]

Equation (6) can generally be written as

\[
x^j G_m(x) = \frac{1}{2} \sum_{k=1}^{j} \left( \frac{j}{k} \right) G_{m-j+k}(x)
\]

For the shifted interval

\[
x^j S_m(x) = \frac{1}{2^j} \sum_{k=1}^{j} \left( \frac{2j}{k} \right) S_{m-j+k}(x)
\]

Where \( S(x) \) is called a shifted Chebyshev polynomials.

Similarly we can write

\[
S_m(x)S_n(x) = \frac{1}{2} \left( S_{m+n}(x) + S_{m-n}(x) \right)
\]

Equation (9) is a numerical approximation.

Next, we shall consider a nonlinear problem whose coefficients and other terms are non-polynomials. The Taylor series expansion for any given function \( f(x) \) at \( x = a \) is given by

\[
f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{m-1}}{(m-1)!}f^{(m-1)}(a) + R_m(x)
\]

\[
= f(a) + \sum_{j=1}^{m-1} \frac{(x-a)^j}{(j)!}f^j(a) + R_m(x)
\]

where

\[
R_m(x) = \frac{(x-a)^m}{m} f^m(\xi), \quad \xi \in (a, x).
\]

### 2.3 Chebyshev Series Representations

This section presents some Chebyshev representations for product of some special functions.

#### 2.3.1 The Chebyshev series approximation for product \( H_m(x) \sin x \)

\[
\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}
\]

\[
G_m(x) \sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!} G_m(x)
\]
\[ x^{2j+1} G_m(x) = \frac{1}{2^{2j+1}} \sum_{k=1}^{2j+1} \binom{2j+1}{k} G_{m-2j+2k-1}(x) \]  
(14)

The finite form of equation (13) can be found from (14) as follows:

\[ G_m(x) \sin x = \frac{1}{2} \sum_{j=0}^{m-1} \sum_{k=0}^{j} \left( -\frac{1}{4} \right)^j \frac{G_{m-2j+2k-1}(x)}{(2j-k+1)!k!} \]  
(15)

For \( n = 1, 2, 3, 4 \) and \( 5 \), we have the Table 1 below

<table>
<thead>
<tr>
<th>( m )</th>
<th>( G_m(x) \sin x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} (G_{m-1}(x) + G_{m+1}(x)) )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} (G_{m-1}(x) + G_{m+1}(x)) )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{48} (-G_{m-3}(x) + 21G_{m-1}(x) + 21G_{m+1}(x) - G_{m+3}(x)) )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{48} (-G_{m-3}(x) + 21G_{m-1}(x) + 21G_{m+1}(x) - G_{m+3}(x)) )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{3840} (G_{m-5}(x) + 75G_{m-3}(x) + 1690G_{m-1}(x) + 1690G_{m+1}(x) - 75G_{m+3}(x) + G_{m+5}(x)) )</td>
</tr>
</tbody>
</table>

2.3.2 The Chebyshev series approximation for product \( G_m(x) \cos x \)

\[ \cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2j!} \]  
(17)

\[ G_m(x) \cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2j!} G_m(x) \]  
(18)

\[ x^{2j} G_m(x) = \frac{1}{2^{2j}} \sum_{k=1}^{2j} \binom{2j}{k} H_{m-2j+2k}(x) \]  
(19)

The finite form of equation (13) can be found from (14) as follows:

\[ G_m(x) \cos x = \sum_{j=0}^{m-1} \sum_{k=0}^{\left\lfloor \frac{j}{4} \right\rfloor} \left( -\frac{1}{4} \right)^j \frac{G_{m-2j+2k}(x)}{(2j-k)!k!} \]  
(20)

For \( n = 1, 2, 3, 4 \) and \( 5 \), we have the Table 2 below
Table 2: Chebyshev Polynomials Equivalence of $G_m(x)\cos x$

<table>
<thead>
<tr>
<th>n</th>
<th>$G_m(x)\cos x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_m(x)$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{8}(-G_{m-2}(x) + 6G_m(x) - G_{m+2}(x))$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{8}(-G_{m-2}(x) + 6G_m(x) - G_{m+2}(x))$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{384}(-G(x)<em>{m-4} - 44G</em>{m-2}(x) + 294G_m(x) - 44G_{m+2}(x) + G_{m+4}(x))$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{384}(-G(x)<em>{m-4} - 44G</em>{m-2}(x) + 294G_m(x) - 44G_{m+2}(x) + G_{m+4}(x))$</td>
</tr>
</tbody>
</table>

3.0 Numerical Experiments

Here we shall consider some numerical experiments as follows:

**Example 1:** The Chebyshev Expansion of $\arccos x$ [6]

Let us consider the Chebyshev expansion of $f(x) = \arccos x$, $f(x)$ is continuous in $[-1,1]$, but is not differentiable at $x = \pm 1$. Observing this, we can expect a noticeable departure from the equioscillation property, as we will see. For this case, the coefficients can be given in explicit form.

**Solution:**

Given that $z_0 = \pi$, $j \geq 1$, then

$$z_j = \frac{2}{\pi} \int_0^\pi \omega \cos j \omega d\omega$$

$$= \frac{2}{\pi} \left\{ \left[ \frac{\omega \sin j \omega}{j} \right]_0^\pi - \int_0^\pi \frac{\sin j \omega}{j} d\omega \right\}$$

$$= \frac{2}{\pi} \left\{ \left[ \frac{\omega \sin j \omega}{j} + \cos j \omega \right]_0^\pi \right\}$$

$$= \frac{2}{\pi} \left( \frac{(-1)^j - 1}{j^2} \right) = \frac{2}{\pi} \left( \frac{(-1)^j - 1}{j^2} \right)$$

From which it follows that
\[ z_{2j} = 0, \quad z_{2j-1} = -\frac{2}{\pi} \left( \frac{2}{(2j-1)^2} \right) \]

We conclude that the resulting Chebyshev expansion of \( f(x) = \arccos x \) is given by

\[ \arccos x = \frac{\pi}{2} G_0(x) - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{G_{2j-1}(x)}{(2j-1)^2} \]

This corresponds with the Fourier expansion

\[ |y| - \frac{\pi}{2} = -\frac{\pi}{4} \sum_{j=1}^{\infty} \frac{\cos(2j-1)y}{(2j-1)^2}, \quad y \in [-\pi, \pi] \]

Notice that the departure from equioscillation close to the endpoints \( x = \pm 1 \), where the function is not differentiable.

**Example 2:**

Let us consider the computation of the Bessel function \( J_0(t) \) in the range \( 0 \leq t \leq 4 \). This corresponds to solving the differential equation for \( J_0(4x) \). But here we shall solve the differential equation of the form

\[ xy'' + y' + 16xy = 0 \quad (21) \]

in the range \( 0 \leq x \leq 1 \) with conditions \( y(0) = 1, y'(0) = 0 \). This is equivalent to the solution of differential equation in \([-1,1]\) because \( J_0(-x) = J_0(x), x \in \mathbb{R} \).

Because \( J_0(4x) \) is an even function of \( x \), \( G_m(x) \) of odd order do not appear in its Chebyshev expansion. By substituting the Chebyshev into the equation we obtain

\[ c_m(xy'') + c_m(y') + 16c_m(xy) = 0, \quad m = 1, 3, 5, \ldots \quad (22) \]

and considering

\[ \frac{1}{2}(c_{m+1} + c_{m-1}) + c_m' + 8(c_{m+1} + c_m) = 0, \quad m = 1, 3, 5, \ldots \quad (23) \]

This equation can be simplified. First, we see that by replacing \( m \to m-1 \) and \( m \to m+1 \) in (22) and subtracting both expressions, we get

\[ \frac{1}{2}(c_{m-2} - c_m) + (c_{m-1} - c_{m+1}) + m(c_{m-1} + c_{m+1}) + 8(c_{m-2} - c_m) = 0, \quad m = 2, 4, 6, \ldots \quad (24) \]

It is convenient to eliminate the terms with the second derivatives. In this way

\[ m(c_{m-1} + c_{m+1}) + 8(c_{m-2} - c_m) = 0, \quad m = 2, 4, 6, \ldots \quad (25) \]

From the Clenshaw’s method we have

Then the following expression can be obtained for the coefficients:

\[ 2mc(s)r = c(s+1)m - 1 - c(s+1)m + 1, \quad m \geq 1 \quad (26) \]

Now, expressions (25) and (26) can be used alternatively in the recurrence process as follows:
\[ c'_{m-1} = c'_{m+1} + 2mc_m \]
\[ c'_{m-2} = c_{m+2} - \frac{1}{8}m(c'_{m-1} + c'_{m+1}) \]
\[ m = N, N-2, N-4, \ldots, 2 \]  \hspace{1cm} (27)

As an illustration, let us take as first trial coefficient \( \tilde{c}_{20} = 1 \), and all higher order coefficient zero. The trial solution of (21) at \( x = 0 \) is given by
\[ \tilde{y}(0) = \frac{1}{2} \tilde{c}_0 - \tilde{c}_2 + \tilde{c}_4 - \tilde{c}_6 + \tilde{c}_8 - \tilde{c}_{10} + \ldots = 8.0509 \times 10^{12} \]  \hspace{1cm} (28)

**Table 3: Computed Coefficients in the Recurrence Processes (27), as we take starting value \( \tilde{c}_{20} = 1 \)**

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \tilde{c}_m )</th>
<th>( \tilde{c}_{r+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.0714 \times 10^{11}</td>
<td>-8.3165 \times 10^{12}</td>
</tr>
<tr>
<td>2</td>
<td>-5.3557 \times 10^{12}</td>
<td>1.3106 \times 10^{13}</td>
</tr>
<tr>
<td>4</td>
<td>2.0045 \times 10^{12}</td>
<td>-2.9303 \times 10^{12}</td>
</tr>
<tr>
<td>6</td>
<td>-2.6772 \times 10^{10}</td>
<td>2.8233 \times 10^{11}</td>
</tr>
<tr>
<td>8</td>
<td>1.8609 \times 10^{10}</td>
<td>-1.5414 \times 10^{10}</td>
</tr>
<tr>
<td>10</td>
<td>-7.7979 \times 10^{9}</td>
<td>-1.5414 \times 10^{10}</td>
</tr>
<tr>
<td>12</td>
<td>2.3281 \times 10^{7}</td>
<td>-1.3549 \times 10^{7}</td>
</tr>
<tr>
<td>14</td>
<td>-4.9280 \times 10^{4}</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>7921</td>
<td>3560</td>
</tr>
<tr>
<td>18</td>
<td>-100</td>
<td>40</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4: Computed Coefficients of the Chebyshev Expansion of the Solution of (21)**

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \tilde{c}_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.1002</td>
</tr>
<tr>
<td>2</td>
<td>-0.6652</td>
</tr>
<tr>
<td>4</td>
<td>0.2469</td>
</tr>
<tr>
<td>6</td>
<td>-0.3325 \times 10^{-1}</td>
</tr>
<tr>
<td>8</td>
<td>-0.2311 \times 10^{-2}</td>
</tr>
<tr>
<td>10</td>
<td>-0.9913 \times 10^{-4}</td>
</tr>
<tr>
<td>12</td>
<td>0.2892 \times 10^{-3}</td>
</tr>
<tr>
<td>14</td>
<td>-0.6121 \times 10^{-7}</td>
</tr>
<tr>
<td>16</td>
<td>0.9838 \times 10^{-9}</td>
</tr>
<tr>
<td>18</td>
<td>-0.1242 \times 10^{-10}</td>
</tr>
<tr>
<td>20</td>
<td>0.1242 \times 10^{-12}</td>
</tr>
</tbody>
</table>
The final values for the coefficients $\tilde{c}_m$ of the solution $y(x)$ of (3.85) will be obtained by dividing the trial coefficients by $\tilde{y}(0)$. This gives the requested values shown in Table 4. The value of $y(1)$ will then be given by

$$y(1) = \frac{1}{2} c_0 + c_2 + c_4 + c_6 + c_8 + \cdots = -0.3971498098638699,$$

(29)

The relative error being $0.57 \times 10^{-13}$ when compared with the value of $J_0(4)$.

4.0 Discussion of Results
The analyzed techniques in conjunction with other properties of Chebyshev polynomial listed in section 2 makes the application of expansion method to a vast class of problems as we can see in Tables 1 and 2 above.

Several questions arise in this successful method. The recursion given in (27) is rather simple, and the exact can be found by expanding $J_0$. The backward recursion scheme for computing the Bessel coefficients are stable as we can see from the above Table 3 and 4 above. In more complicated recursion schemes this information is not available. The scheme may be of large order and may have several solutions of which the asymptotic behavior is unknown. So, in general, we don’t know if Clenshaw’s method for differential equations computes the solution that we want, and if for the wanted solution the scheme is stable in the backward direction. The above results are obtained using MATLAB Program.

5.0 Conclusion
The technique that makes expansion method applicable to problems with polynomial and non-polynomial terms has been studied. The analyzed techniques in conjunction with other properties of Chebyshev polynomial listed in section 2 makes the application of expansion method to a vast class of problems. We conclude that expansion method has been extended to cover a wider range of problems involving polynomial and non-polynomial coefficients differential equations.

References