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The problem of locating an obstacle for the fundamental eigenvalue of laplacian and p-laplacian

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Introduction générale

The problem of locating an obstacle to the fundamental intrinsic value is to locate the position of the setting up barriers or wells to maximize or minimize the first eigenvalue of the operator considered.
In [22] , the authors studied this problem by considering the Laplace operator or Schrodinger
In [32] Long-Jiang Gua, Xiaoyu Zengb, and Huan-Song Zhoub have studied the existence of asymptotic behavior of the base states for the eigenvalue problem of the p-laplacian .
In [31] Leandro , Del Pezzo and Julio Studied the first eigenvalue for the p-Laplacian operator with the boundary conditions of Dirichlet and Neumann (mixed boundary conditions).
In [14] Daniele Valtorta gave the estimate of the first non-trivial eigenvalue of the p-Laplacian on a compact Riemannian manifold with a non-negative Ricci curvature and characterize the case of equality. He studied the following problem:

$$\left\{ \begin{array}{lcll} \Delta_p(u) & = & \lambda_{1,p}|u|^{\alpha-2}u & \text{on} \quad \Omega \\ \langle \nabla u \; n \rangle & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{1}$$

Daniele Valtorta has proved the following strong estimate:

$$\frac{\lambda_{1,p}}{p-1} \geqslant \frac{\Pi_p^p}{d^p}$$

With

$$\Pi_p = \int_{-1}^{1} \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin(\frac{\pi}{p})}$$

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In the first chapter we will study the variation of the fundamental eigenvalue according to the position of obstacl
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In this article we will study the variation of the fundamental value following the clean obstacl position.
Let D is open bounded in $\mathbb{R}^N$ and B is obstacle moving at inside D.
We will study the variation of λ_1. the first eigenvalue of the operator $-\Delta$ if the obstacle B moves inside D.
The approach to the study of problems is as follows:
We will pose the problem. So we study the derivation and the variation in λ the first eigenvalue of the Laplace operator.
The variation of Ω is explained by the fact that B moves in D without going out. If B is hard obstacle,the movement of B in D is done either by translation or by rotation or combining these two types of movement. If B is considered a Soft obstacle, B can be transformed by dilation.
We gave the derivative of the first eigenvalue of the Laplace operator for a hard obstacle and in the case of a soft obstacle or a well.
We will study the variation of the first eigenvalue of the Laplace operator λ. and we also state a theorem on the variation of λ, that will give us the obstacle position for λ is minimal

In the second chapter In the second chapter of this book, we will study the same problem studied in the first chapter by using the same techniques used in our paper [2] .
In [22], the authors studied this problem by considering the Laplace or Schrodinger operator defined within a fixed, bounded, open domain D with zero Dirichlet boundary conditions. Inside this domain, they placed a ball which represents an obstacle or a well , the position of which is under their control, and their goal was to locate the optimal position of the piece under their control. And in their works (cf [22]), one can find some interesting partial answers assuming convexity and/or symmetry properties for D. They also gave illustrative examples.
In this part of our work, considering an obstacle or a well not necessarily a ball, we study sufficient conditions to obtain the minimum or maximum value for the first eigenvalue of the Laplace or Schrodinger operator.
In the third chapter of this book, we will study the obstacle positron problem for the p-Laplacian operator.
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istence of asymptotic behavior of the base states for the eigenvalue problem of the following p-laplacian equation:

$$\Delta_p u = V(x)|u|^{p-2}u = \mu|u|^{p-2}u + a|u|^{s-2}u, \;\; x \in \mathbb{R}^N$$

with $p \in (1,n)$, $s = p + \frac{p^2}{n}, a \geqslant 0$ and $\mu \in \mathbb{R}$ Is a parameter and $V(x)$ Is a field of vectors satisfying certain assumptions.
In [31] Leandro , Del Pezzo and Julio Studied the first eigenvalue for the p-Laplacian operator with the boundary conditions of Dirichlet and Neumann (mixed boundary conditions). They considered the following problem:

$$\begin{cases} \Delta_p u = & \lambda\alpha|u|^{\alpha-2}u|v|^{\beta} \quad \text{on} \quad \Omega \\ \Delta_q u = & \lambda\beta|u|^{\alpha}u|v|^{\beta-2}v \quad \text{on} \quad \Omega \end{cases} \tag{2}$$

with $\frac{\alpha}{p} + \frac{\beta}{q} = 1$ and Next mixed boundary conditions:

$$u = 0, \;\; |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} \;\text{ sur }\; \partial\Omega$$

In [33] idrissa ly studied the behavior of the first eigenvalue of the p-Laplacian operator $\lambda_1^p(\Omega_n)$ avec la condition du Dirichlet homogène au bord du domaine variable (Ω_n) ,où (Ω_n) est une famille séquentielles des perturbations géométrie.
In [14] Daniele Valtorta gave the estimate of the first non-trivial eigenvalue of the p-Laplacian on a compact Riemannian manifold with a non-negative Ricci curvature and characterize the case of equality. He studied the following problem:

$$\begin{cases} \Delta_p(u) & = & \lambda_{1,p}|u|^{\alpha-2}u & \text{on} & \Omega \\ \langle \nabla u \; n\rangle & = & 0 & \text{on} & \partial\Omega \end{cases} \tag{3}$$

Daniele Valtorta has proved the following strong estimate:

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In this article we will study the obstacle position problem for the p-Laplacian operator.
The obstacle locating problem for the fundamental eigenvalue is to locate the position of the obstacle placement so as to maximize or minimize the eigenvalue of the p-Laplacian operator. We were interested in the following problems:

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Let Ω be a bounded open set of $\mathbb{R}^N$ and K An obstacle that moves The interior of D. We consider the problem :

$$\left\{ \begin{array}{rcll} \Delta_p(u) & = & \lambda_{1,p}|u|^{\alpha-2}u & \text{in} \quad D\backslash K \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{4}$$

Let $\lambda_{1,p}$ The first eigenvalue of the p-Laplacian operator with certain hypotheses we want to give the necessary and sufficient conditions so that the first eigenvalue of the p-Laplacian operator is minimal or we want to determine the position of K in Ω so that $\lambda_{1,p}$ Is minimal where $\lambda_{1,p}$ Represents the first eigenvalue of the p-Laplacian operator

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Chapter 1

The variation of the first eigenvalue of the Laplace operator and The problem of locating an obstacle

The problem of locating an obstacle to the fundamental intrinsic value is to locate the position of the setting up barriers or wells to maximize or minimize the first eigenvalue of the operator considered.
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We will pose the problem. So we study the derivation and the variation in λ the first eigenvalue of the Laplace operator.
The variation of Ω is explained by the fact that B moves in D without going out. If B is hard obstacle,the movement of B in D is done either by translation or by rotation or combining these two types of movement. If B is considered a Soft obstacle, B can be transformed by dilation.
We gave the derivative of the first eigenvalue of the Laplace operator for a hard obstacle and in the case of a soft obstacle or a well.
We will study the variation of the first eigenvalue of the Laplace operator λ. and we also state a theorem on the variation of λ, that will give us the obstacle position for λ is minimal

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1.1 presentation of different obstacle and first eigenvalue of the Laplace operator

Let D is open fixed in $\mathbb{R}^N$ and B is obstacle moving at inside D.
In this work we study the minimization of the first eigenvalue of the operator Laplace Dirichlet.
Specifically, placing B to inside D with boundary conditions of Dirichlet zero on the border of $\Omega = D/B$
we want to determine the position of B in D for λ_1 is minimal

$$\begin{cases} -\Delta u = \lambda_1 u & \text{in } D \backslash B = \Omega \\ u = 0 & \text{on } \partial(D\backslash B) = \partial\Omega \end{cases} \tag{1.1}$$

Define a vector field

$$\begin{aligned} V : \mathbb{R}^N & \longmapsto \mathbb{R}^N \\ x & \longmapsto (V_1(x),\ V_2(x),\ V_3(x)..., V_N(x)) \end{aligned}$$

for every real t small enough, it identifies areas disturbed:

$$\Omega_t = (Id + tV)(\Omega) = \{x + tV(x), \quad x \in \Omega\}.$$

The variation of Ω is explained by the fact that B moves in D without going out. If B is hard obstacle,the movement of B in D is done either by translation or by rotation or combining these two types of movement.
If B is considered a Soft obstacle, B can be transformed by dilation.
After the disturbance problem (1.1) becomes:

$$\begin{cases} -\Delta u_t & = & \lambda\ u_t \text{ in } \Omega_t \\ u_t & = & 0 \text{ on } \partial\Omega_t \end{cases}$$

With $\Omega_t = (Id + tV)(\Omega) = \{x + tV(x), \quad x \in \Omega.\}$
The derivative of is given by

$$\begin{cases} -\Delta u' = \lambda_k u' + \lambda_k' u \text{ in } \Omega \\ u' = -\dfrac{\partial u}{\partial n} V.n \text{ on } \ \Gamma : \displaystyle\int_\Omega u u' dx = 0 \end{cases} \tag{1.2}$$

with $n(\centerdot)$ external unit normal to $\partial\Omega$.
We will give the definition of spectrum and the first eigenvalue of the operator Laplace Dirichlet.

Definition 1.1.1 *Let A square symmetric matrix of order N of $M(\alpha, \beta, \Omega)$. constant λ is a eigenvalue of the operator*

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$\mathcal{A} = -div(A\nabla)$ *with Dirichlet boundary conditions* Ω *if* $u \neq 0$ *is the solution of problem*

$$\begin{cases} \mathcal{A}u = \lambda u \quad dans\ \Omega \\ u = 0 \quad sur\ \partial\Omega \end{cases} \tag{1.3}$$

The function u *is called proper function of* $\mathcal{A}$ *Associated with the eigenvalue* λ. *The set of eigenvalue is called the spectrum of* $\mathcal{A}$. *Let* $\sum(\mathcal{A})$ *this set.*

Definition 1.1.2 *Let* $\alpha, \beta \in \mathbb{R}$ *such as* $0 < \alpha < \beta$. *We denote by* $M(\alpha, \beta, \Omega)$ *all square matrices of order* N *as*

$$A = (a_{ij})_{1\leq i,j\leq n} \in (L^{\infty}(\Omega))^{N\times N}$$

satisfying

$$(i)\ A(x)\xi \cdot \xi \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^N \quad in\ \Omega \tag{1.4}$$

$$(ii)\ |A(x)\xi| \leq \beta|\xi| \tag{1.5}$$

Proposition 1.1.1 *the Problem of minimization following :*

$$\lambda_1 = min\Big\{\int_{\Omega} A\nabla u \nabla u,\ u \in H_o^1(\Omega)\setminus\{0\} \int_{\Omega} u^2 dx = 1\Big\}.$$

has a solution.

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Proof of proposition 1.1.1

The set of eigenvalues of $\mathcal{A}$ is the set

$$G=\left\{a\in\mathbb{R}\text{ such as }a=\int_\Omega A\nabla u\nabla u\text{ avec }u_o\in H_o^1(\Omega)\setminus\{0\}\text{ et }||u||_{L^2(\Omega)}=1\right\}$$

let

$$J(u)=\int_\Omega A\nabla u\nabla u\;dx.$$

We show that the function $J(u)$ admits a lower bound K with

$$K=\left\{u\in H_o^1(\Omega),\;||u||_{L^2(\Omega)}=1\right\}$$

$$A\in\mathcal{M}(\alpha,\beta,\Omega)\Longrightarrow(A\;\nabla u\,\centerdot\,\nabla u)\geq\alpha|\nabla u|^2\geq 0$$

$$\Longrightarrow\int A\nabla u\,\centerdot\,\nabla u)\geq 0\quad\forall u\in H_o^1(\Omega)$$

(i) $J(u)$ admits a lower bound. so G admits a lower bound.

Let

$$\alpha=inf\;G=\inf_{\substack{u^\varepsilon\in H_o^1(\Omega)\\ ||u||_{L^2(\Omega)}\neq 0}}J(u)=\inf_{u\in K}J(u)$$

(ii) $\lambda_1\in G\Longrightarrow\alpha\leq\lambda_1$

Using the definition of lower bound $\exists u_n\in H_o^1(\Omega)/\{0\}$ such as

$$J(u_n)\longrightarrow\alpha.$$

$J(u_n)=\displaystyle\int(A^{\mathcal{E}}\nabla u\,\centerdot\,\nabla u)$ convex, continuous weakly sequentially s.c.i.

u_n admits a lower bound $H_o^1(\Omega)$, moreover $H_o^1(\Omega)$ is reflexive $\Longrightarrow\exists u_n^\epsilon$ a subsequence of u_n such as $u_n^\epsilon\longrightarrow u$ or K closed J weakly sequentially s.c.i.

$$\Longrightarrow J(u_n^\epsilon)\leq\underline{lim}\;J(u_n^\epsilon)=\alpha$$

$$u_n^\epsilon\in K\Longrightarrow J(u_n^\epsilon)\geq\inf_{u_n^\epsilon\in K}J(u)=\alpha$$

$$J(u_n^\epsilon)=\inf_{u_n^\epsilon\in K}J(u)=\min_{u_n^\epsilon\in K}J(u_n^\epsilon)$$

then $J(u_n^\epsilon)=\alpha=min\;G$

$$\alpha=min\;G=J(u_n^\epsilon)=\frac{\displaystyle\int_\Omega A\nabla u\;\nabla u}{\displaystyle\int_\Omega(u_o)^2\;du}$$

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thus α is a eigenvalue of A. λ_1 this is the smallest eigenvalue A.

(iii) $\Longrightarrow$ $\lambda_1 \leq \alpha$ Using (i), (ii) and (iii)

$$\lambda_1 = \alpha = \min_{\substack{u \in H_o^1(\Omega) \\ u \neq 0}} J(u)$$

thus

$$\lambda_1 = \min_{\substack{u \in H_o^1(\Omega) \\ u \neq 0}} \Big(\frac{\displaystyle\int_\Omega A\nabla u \ \nabla u \ dx}{\displaystyle\int_\Omega u^2 \ dx} \Big)$$

■

We will give some definitions before formulating more precisely the problem.

Definition 1.1.3 *It said that the obstacle B is Soft,if the operator we are going to consider is of the following form*
- $\Delta + \alpha\chi_B$ *ou* $\alpha > 0$ *and* χ_B *is the indicator function of the region* B.
A hard obstacle corresponds to $\alpha = +\infty$ *and they say* B *is well if* α *is negative.*

In the case of a hard obstacle:
Define for any real t pretty small $T_t(B)$ as translation, rotation or a face.
ask $J_2(\Omega_t) = \displaystyle\int_{\Omega_t} dx - v_o$ with $v_o > 0$ and
$\Theta_\varepsilon = \Big\{ \Omega_t = D \backslash T_t(B),$ ouvert de $\mathbb{R}^N$ and verifying ownership ε
cone and $\displaystyle\int_{\Omega_t} dx = v_o \Big\}$ So the problem becomes determining the position of B such than

$$\min_{\Omega_1 \in \Theta_\varepsilon} \lambda_1(\Omega_t) \text{ is reached}$$

$$\text{or } \lambda_1(\Omega_t) = \min_{u \in H_o^1(\Omega)} \Big\{ \int_{\Omega_t} |\nabla u|^2 dx \ : \ \int_{\Omega_t} u^2 dx = 1, \Big\}$$

In the case of a Soft obstacle is removed the constraint on the volume of all eligible Θ_ε. Let $V_\varepsilon = \Big\{ \Omega_t = D \backslash T_t(B)$ open from $\mathbb{R}^N$ verifying ownership
$\varepsilon -$ cone$\Big\}$ or T_t may be a dilation where a composition of a translation and a dilation then the problem is to determine position Ω_t so that $\min\limits_{\Omega_t \in V_\varepsilon} \lambda_1(\Omega_t)$ is reached
or $\lambda_1(\Omega_t) = \min\limits_{u \in H_o^1(\Omega)} \Big\{ \displaystyle\int_D |\nabla u|^2 + \alpha \int_D \chi_B u^2 \ , \ \int_D u^2 = 1 \text{ or } \alpha \in \mathbb{R} \Big\}$

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1.2 The variation of the first eigenvalue and locating of obstacle

We gave the derivative of the first eigenvalue of the Laplace operator for a hard obstacle and in the case of a soft obstacle or a well.
We will study the variation of the first eigenvalue of the Laplace operator λ. and we also state a theorem on the variation of λ.

Definition 1.2.1 *Let J a functional set to Ω. Called derivative (Gateaux) of J on point Ω, the direction of deformation V the limit denoted $dJ(\Omega, V)$, if it exists*

$$dJ(\Omega, V) = \lim_{t \longmapsto 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

Definition 1.2.2 *Called derivative form λ the limit denoted λ' if it exists*

$$\lambda' = \lim_{t \longrightarrow 0} \frac{\lambda(\Omega_t) - \lambda(\Omega)}{t}$$

This definition can be found in [24].

Proposition 1.2.1 *Let Ω a open bounded of class C^2, we suppose that $\lambda_k(t)$ is eigenvalue simple. So functions*

$$t \longmapsto \lambda_k(t) \text{ et } t \longmapsto u_t \in L^2(\mathbb{R}^N)$$

are differentiable in $t = 0$ et $u' \in H^1(\Omega)$ is the only solution to (1.2) with

$$\lambda_k'(0) = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 V.n \tag{1.6}$$

Proof

We are going to give a sketch of proof by giving some hints giving the desired result. To prove this we use in part the implicit functions theorem. We also use the shape derivative techniques see for instance pionner works of M. Schiffer [37] or [24], [36]. Let us give now some hints for the proof:
Let us consider the problem

$$\begin{cases} -\Delta u_\Omega = \lambda_\Omega u_\Omega & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

Using the shape derivative, we get

$$\begin{cases} -\Delta u' = \lambda' u + \lambda u' & \text{in } \mathcal{D}(\Omega) \\ u' = -\frac{\partial u}{\partial n} V(0).n & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

Multiplying by u the first equation of the above system, using the Green formula and finally replacing u' by its value on the boundary of Ω, we get:

$$\lambda_k'(\Omega, V) = -\int_{\partial K} (\frac{\partial u}{\partial n})^2 V(0).n d\sigma$$

■

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We gave the derivative of the first eigenvalue of the Laplace operator for a hard obstacle and in the case of a soft obstacle or a well. We shall study in the following proposition:

Proposition 1.2.2 *Consider the case of obstacle soft or a well so using [?] we have the following problems*

$$\begin{cases} -\Delta u + \alpha\chi_B(x)u & = \lambda u \text{ in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

With $\alpha \in \mathbb{R}$ *and* χ_B *is the indicator function of the region* B*. with border* B *is smooth by piece.*
Suppose that B *can be displaced by a distance in the direction of a vector field* V
So we:

$$\frac{d\lambda}{dV} = \alpha \int_{\partial B} |u|^2 n.V \; ds \tag{1.9}$$

proof : see [22]

We will study the variation of the eigenvalue λ. We recall a useful definition which will allow us to make constructions on the domain Ω , and we enunciate a theorem on the variation of λ.

Definition 1.2.3 *Let* P *a hyperplane of dimension* $N-1$ *which intersects* Ω*.*
For any connexe set S *didn't intersects* P*,*
We call S^P *symmetrical with respect to* P*.*
They say the domain Ω *is possessed the property the inner reflection compared to* P *if there is a connexe component* Ω_S *of* $\Omega\backslash P$ *such as* Ω_S^P *is a sub - set own another connexe component* Ω_b *of* $\Omega\backslash P$*. such* P *will be called hyperplane of inner reflection for* Ω *, with,* Ω_S *will be called the small side for* Ω *(and* Ω_b *will be called the big side for* Ω*).*

we enunciate a theorem on the variation of the first eigenvalue of the Laplace operator in the different cases of obstacle
This theory was demonstrated in [22] we'll show ca using other technical

Theorem 1.2.1 *suppose that* Ω *is possessed the property the inner reflection compared to* P *with* B *a ball.*
suppose that B *moves with a translational movement following a field vector* V *in particular* P *in the same direction with* V *and pointing of the small side to big side.*

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Let λ_1 is the first eigenvalue of the Laplace Dirichlet:

$$\begin{cases} -\Delta u = \lambda_1 u & in\ D\backslash B = \Omega \\ u = 0 & on\ \partial(D\backslash B) = \partial\Omega \end{cases} \tag{1.10}$$

So in the case of a hard or soft obstacle

$$d\lambda_1(\Omega, V) > 0$$

in the case of a well

$$d\lambda_1(\Omega, V) < 0$$

before giving the proof of this theorem, we will give some useful results for the different stages of proof

Proposition 1.2.3 *: [Maximum principle for self-adjoint operators]*
Let Ω domain open , regular of $\mathbb{R}^N$. Consider an elliptic operator of second order in Ω

$$L = \partial_i\ (a_{ij}(x_i\ \partial_j) + c(x)$$

such as

$$c_o|\xi|^2 \leq a_{ij}(x)\ \xi_i\xi_j \leq C_o|\xi|^2 \quad c_o, C_o > 0\ ,\ \forall\xi \in \mathbb{R}^N$$

with $a_{ij}(x) \in C(\Omega)$, $c(x) \in L^\infty(\Omega)$.
Then we have the maximum principle by :

$$(PM)' : \begin{cases} u \in H^1(\Omega) \\ Lu \leq 0 \\ u|_{\partial\Omega} \geq 0 \end{cases} \quad is\ verifie\ \Longrightarrow\ u \geq 0\ \ pp\ \ in\ \ \Omega$$

Proposition 1.2.4 *Let λ_1 the first eigenvalue of the operator $-L$*
So $(PM)'$ is checked if and if $\lambda_1 > 0$.

Proof of Theorem 1.2.1
There are three cases to consider an obstacle on a soft obstacle and a well.We consider the obstacle hard to the last Position,for the other two cases we explain that for any point x of ∂B which is on the small $\Omega = D|K$ such as $u(x) < u(x^p)$ and for the case of an obstacle hard we consider on the formula of Hadamard see [22] Suffice it to prove that $|\nabla u(x)| < |\nabla u(x^p)|$.
Ω has the property of reflection following a hyperplane T_λ of the equation $(x_N = \lambda)$ in the direction of the axis x_N suppose $B = K$.

- D admits an axis of symmetry of equation $(x_N = 0)$.
- K_λ^+ the part of K located further above of T_λ.

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- $\sigma_\lambda(K_\lambda^+)$ the symmetrical of K_λ^+ by compared to T and $K_\lambda = K|K_\lambda^+$
- D_λ^+ the part of Ω completely located above of T_λ
- $\sigma_\lambda(D_\lambda^+)$ the symmetrical of D_λ^+ compared to T_λ and $D_\lambda^- = \Omega \setminus \{D_\lambda^+ \cup \sigma_\lambda(D_\lambda^+)\}$

one may encounter the following two cases of figures : the figure that explains the notation

- (i) $\sigma(K_\lambda^+)$ is inner tangent to $\partial\Omega$ at a point y_o, with $y_o \notin T_\lambda$
- (ii) T is orthogonal to $\partial\Omega$ at a point x_o that is to say $\exists\lambda$ such as $\sigma_\lambda(\Omega_\lambda^+) \subset \Omega$.
- Let the case (i) is present .
- Let the case (ii) is present.

Suppose $\lambda \leq 0$ to fix ideas $\Sigma_\lambda = K_\lambda^+ \cup \sigma(K_\lambda^+)$. Let v the function defined on Σ_λ by $\forall x \in \Sigma_\lambda$
$v(x) = u(x^p)$ with $x^p = \sigma_{\lambda_o}(x)$. so on Σ_λ we have

$$\begin{cases} -\Delta u = \lambda_1 u & in \quad K_\lambda^+ \cup \sigma(K_\lambda^+) \\ u = 0 & on \quad \partial K_\lambda \setminus T_{\lambda_o} \\ u(x) = u(x^p) & on \quad T_{\lambda_o} \\ u(x) = 0 & on \quad \partial\sigma(K_\lambda^+) \setminus T_{\lambda_o} \end{cases}$$

$$\begin{cases} -\Delta(u(x) - v(x) = \lambda(u(x) - v(x)) & in \quad K_\lambda^+ \cup \sigma(K_\lambda^+) \\ u(x) - v(x) = -u(x^p) & on \quad \partial K_\lambda^+ \ / \ T_{\lambda_o} \\ u(x) - v(x) = 0 & on \quad T_{\lambda_o} \\ u(x) - v(x) = -u(x^p) & on \quad \partial\sigma(K_\lambda^+) \setminus T_{\lambda_o} \end{cases} \tag{1.11}$$

determining of sign u in Ω we will determine the sign of u using the maximum principle (PM) and the proposition (1.2.4)

Let $a_{ij}(x) = I \quad c(x) = 0$

a_{ij} elliptical $\quad a_{ij} \in C(\Omega) \;\; c(x) \in L^\infty(\Omega)$.

Let $L = \partial i \;\; (a_{ij}(x)\ \partial_j) + c(x)$.

so $L(u) = \Delta u$. consequently the first eigenvalue of $-L$ is equal to λ_1 or $\lambda_1 > 0$. So using to the proposal(1.2.4) $(PM)'$ is satisfying $\Longrightarrow u \geq 0$ pp in Ω.

consequently the equation (1.11) becomes

$$\begin{cases} \Delta(v(x) - u(x)) + \lambda(v(x) - u(x)) = 0 & in \quad \Sigma_\lambda \\ v(x) - u(x) = u(x^p) \geq 0 & sur \quad \partial K_\lambda^+ \setminus T_{\lambda_o} \\ v(x) - u(x) = 0 & sur \quad T_{\lambda_o} \\ v(x) - u(x) = u(x^p) \geq 0 & sur \quad \partial\sigma(K_\lambda^+) \setminus T_{\lambda_o} \end{cases}.$$

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Let $\omega = v(x) - u(x)$.
So Using to the maximum principle if we pose

$c(x) = \lambda$

$L\omega = \Delta\omega + \lambda\omega$ our $a_{ij} = I$

$L = \partial_i(a_{ij}(x)\ \partial_j) + \lambda$

I elliptical $c(x) \in L^\infty(\Omega)$

$$\Longrightarrow \begin{cases} L\omega = 0 \quad on \ \ \Sigma_\lambda \\ \omega \geq 0 \quad on \quad \partial\Sigma_\lambda \end{cases}.$$

Using $(PM) \Longrightarrow \omega \geq 0\ pp$ in $\sum_\lambda \Longrightarrow \omega(x) = u(x^p) - u(x) \geq 0\ pp$ in $\sum_\lambda$ we will show that

$$\omega(x) \neq 0 \ \ \forall x \in \partial K_\lambda^+ \backslash T_\lambda.$$

if $x \in K_\lambda^+ \backslash T_\lambda$. we pose $u(x) = u(x^p) \Longrightarrow \omega(x) = 0$.

so $\begin{cases} L\omega = 0 \quad in\ \Sigma_\lambda \\ \omega = 0 \quad \text{on } \partial\Sigma_\lambda \end{cases} \Longrightarrow$ using $(PM)'$ $\omega \geq 0$ in Σ_λ.

we pose $h = -\omega$

$\begin{cases} L(h) = 0 \quad in\ \Sigma_\lambda \\ h = 0 \qquad on\ \partial\Sigma_\lambda \end{cases} \Longrightarrow$ using $(PM)'$ $h \geq 0$ on $\partial\Sigma_\lambda \Longrightarrow -\omega \geq 0$.

$\Longrightarrow \omega \leq 0 \Longrightarrow \omega = 0$ pp in Σ_λ let $x \in \Sigma_\lambda$

or $\dfrac{\partial\omega(x)}{\partial n} = \lim\limits_{h\to 0} \dfrac{\omega(x - nh) - \omega(x)}{h} = 0.$

so $\dfrac{\partial\omega(x)}{\partial n} = 0$

or $u(x) - v(x) = 0$ we apply Lemma (1) page (308) [36].

we have $\dfrac{\partial\omega(x)}{\partial n} > 0$ which is contradictory.

So if $x \in \partial K_\lambda^+ \setminus T_{\lambda_o}$ $\quad u(x^p) - u(x) > 0$

$\Longrightarrow u(x^p) > u(x)$.

In the case of an hard obstacle we use the formula of Hadamard this time we consider the function $\omega(x) = u(x) - u(x^p)$ according to the $(PM)'$ on a $\omega(x) < 0$ inside of $K_\lambda^+$to complete the proof in this case we use the lemma (1) page (308) [36]. So the derivative normal of $\omega(x)$ is positive or if the second derivative of $\omega(x)$ in that direction, the second is impossible considering the equation of eigenvalue from where $|\nabla u(x)| < |\nabla u(x^p)|$ for any $x \in \partial\ K_\lambda^+ \setminus T_{\lambda_o}$ ■

Conclusion

Using the property of reflection, concepts of small and big side of the domain Ω and the variation of λ_1 .
we find that λ_1 is strictly increasing when the obstacle B is placed in contact with

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the border towards the large side. Thus λ_1 is minimal when the obstacle touches the boundary of the domain Ω. ■

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Chapter 2

The problem of obstacle for the fundamental eigenvalue

Studying problems of Analysis of Stability of the Exterior and Interior Bernoulli's Free Boundary Problems form in [2] leads us to get interesting information about obstacle problem for the principal eigenvalue (the first eingenvalue of the Laplace operator with Dirichlet boundary conditions). So what is the obstacle problem?
The problem of locating an obstacle to the fundamental eigenvalue value is to locate the setting up position of the barriers or wells in order to maximize or minimize the first eigenvalue of the considered operator .
In [22], the authors studied this problem by considering the Laplace or Schrodinger operator defined within a fixed, bounded, open domain D with zero Dirichlet boundary conditions. Inside this domain, they placed a ball which represents an obstacle or a well , the position of which is under their control, and their goal was to locate the optimal position of the piece under their control. And in their works (cf [22]), one can find some interesting partial answers assuming convexity and/or symmetry properties for D. They also gave illustrative examples.
In this part of our work, considering an obstacle or a well not necessarily a ball, we study sufficient conditions to obtain the minimum or maximum value for the first eigenvalue of the Laplace or Schrodinger operator. The beginning of this study is that, we suppose we have critical position of the obstacle in the domain. This means that the shape derivative of the fundamental eigenvalue which we shall explain in this paper, equals a constant on the boundary of the considered domain.
In the following , we focus our efforts on the minimization problem because the maximization one uses the same techniques. The problem of the optimal placement of the obstacle is stated as follows: We assume that $\Omega = D\backslash K$, where $K \subset D$, represents an obstacle or well which is a $\mathcal{C}^2$-regular domain. The shape of K is fixed a priori and only its position changes by rigid motions (translaions and rotations, but in this study we are going to focus on motions by translations). Let us suppose

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that Ω is a critical point, i.e we have a position of K such that the derivative with respect to the domain of the first eigenvalue is equal to a constant on the boundary of K or D.
Our aim is to give sufficient conditions to characterize the shape of the obstacle K, so that the fundamental eigenvalue of the Laplace or Schrodinger operator on Ω with Dirichlet condition on the boundary of the domain is minimum. The obstacles we shall consider may be hard, i.e. the zero Dirichlet conditions are additionally imposed on the boundary of K, or they may be soft, that is the operator we are going to consider is of the following form:

$$-\Delta + \alpha\chi_K I.$$

where $\alpha \in \mathbb{R}$, and χ_K is the indicator function of the region K defined by:

$$\chi_K(x) = \begin{cases} 1 & \text{if} \quad x \in K \\ 0 & \text{if} \quad x \notin K \end{cases} \tag{2.1}$$

A hard obstacle corresponds to $\alpha = +\infty$. The term of well refers to the case where the constant α is negative. These types of operators are defined in standard ways, and our sign convention of the fundamental eigenvalue with a hard obstacle is positive and in the case of well it may be negative. For more information on these operators, see for instance [17].
Notation
In this paper, we are going to use the following notation

$$V(x,0) = V(0) = V.$$

The placement obstacle problem for the fundamental eigenvalue

In this paper, we mention that the same techniques are used in our paper [2]. We will study the obstacles positions problem using findings from [2] on the functional J introduced by H. Alt W and L. Caffarelli in [1].
We offer the following details. Suppose we have a critical point for the first eigenvalue of the Laplace operator, we will give the quadratic form associated with the first eigenvalue of the Laplace operator and we will conclude by giving the placement obstacle problem.

2.1 Critical point

Let us define:

$$\mathcal{O}_\epsilon = \{w \subset D,\ \ w \text{ open set verifying the uniform cone property and } vol(w) = m_0\}$$

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We assume that $\omega = D \backslash K$, where $K \subset D$ represents an obstacle which is a $\mathcal{C}^2$-domain. The shape of D and the obstacle K are fixed a priori and only the position of K changes .

Proposition 2.1.1 *(Hard obstacle case)*
Let us consider the following problem:

$$\left\{ \begin{array}{rcll} -\Delta u_\Omega & = & \lambda_\Omega u_\Omega & in \quad \Omega \\ u & = & 0 & on \quad \partial\Omega \end{array} \right. \tag{2.2}$$

where $\Omega = D \backslash K$ and K is a $\mathcal{C}^2$ domain.
The first eigenvalue is defined by:

$$\lambda_\Omega = \inf\{\int_\Omega |\nabla u|^2 dx : u \in W_0^{1,2}(\Omega) / \int_\Omega u^2 dx = 1\}.$$

.
Assuming that there is $\Omega \in \mathcal{O}_\epsilon$., we have
$\lambda'(\Omega, V) = -\int_{\partial K} (\frac{\partial u}{\partial n})^2 \; V(0).n \; d\sigma$.
And if Ω is a critical point for λ_Ω, then there is a Lagrange multiplier β_Ω such that :

$$-(\frac{\partial u}{\partial n})^2 = \beta_\Omega \; on \; \partial K$$

n being the exterior unit normal vector to $\Omega = D \backslash K$ (n is the interior unit normal vector to K).

Proof
We are going to give a sketch of the proof through some hints giving the desired result. The proof uses in part the implicit functions theorem. We also use shape derivative techniques, see for instance pionner works of M. Schiffer [37] or [24], [36]. Let us give some hints for the proof:
Consider the problem

$$\left\{ \begin{array}{rcll} -\Delta u_\Omega & = & \lambda_\Omega u_\Omega & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{2.3}$$

Using the shape derivative, we get

$$\left\{ \begin{array}{rcll} -\Delta u' & = & \lambda' u + \lambda u' & \text{in} \quad \mathcal{D}(\Omega) \\ u' & = & -\frac{\partial u}{\partial n} V(0).n & \text{on} \quad \partial\Omega \end{array} \right. \tag{2.4}$$

Multiplying the first equation of the above system by u, using the Green formula and finally replacing u' by its value on the boundary of Ω, we get:

$$\lambda'(\Omega, V) = -\int_{\partial K} (\frac{\partial u}{\partial n})^2 V(0).n d\sigma$$

The Lagrange multiplier appears because of the constraint required on the volume of Ω. ∎

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Proposition 2.1.2 *(Well case)*
Consider the following problem:

$$\begin{cases} -\Delta u_\Omega + \alpha\chi_K u_\Omega &= \lambda_\Omega u_\Omega \quad in \quad D \\ u &= 0 \quad\quad on \quad D \end{cases} \tag{2.5}$$

where $\Omega = D\backslash K$ *and* K *is a* $\mathcal{C}^2$ *domain .*
The first eigenvalue is defined by:

$$\lambda_\Omega = \inf\{\int_D |\nabla u|^2 dx + \alpha \int_D \chi_K u^2 dx : u \in W_0^{1,2}(\Omega) \int_D u^2 dx = 1\}.$$

Then, we have:
$\lambda'(\Omega, V) = \alpha \displaystyle\int_{\partial K} u^2 V(0).n d\sigma$, *where* n *is the exterior unit normal to* K.
And if

$$\Omega \in \mathcal{O} = \left\{\omega \subset D, K \subset \omega \;\; w \;\; open\ set\ verifying\ the\ \epsilon - cone\ property\ and\ \int_\omega \chi_K dx = vol(K) = m_0\right\}$$

(m_0 *being a fixed positive real number) is a critical point for* λ_Ω*, then there exists a Lagrange multiplier* γ_Ω *such that :*

$$\alpha u^2 = \gamma_\Omega \ on\ \partial K$$

If Ω *is a critical point for* λ_Ω *and if, moreover, the volume of* K *may change then* $u = 0$ *on* ∂K.

To prove this proposition, one uses the same techniques and the same steps in the proof for the hard obstacle the case. However, the equation satisfied by the shape derivative changes as follows:

$$\begin{cases} -\Delta u' + \alpha\chi_K u' + \alpha\chi_K u V(0).n_e &= \lambda' u + \lambda u' \quad \text{in} \quad \mathcal{D}(D) \\ u' &= 0 \quad\quad\quad\quad \text{on} \quad \partial D \end{cases} \tag{2.6}$$

2.2 Quadratic form associated with the obstacle placement problem

The quadratic shape is obtained by calculating the second derivative of $\lambda'(\Omega, V)$ against the domain. Let us take V given by $V(x;t) = v(x)n(x)$, $v \in H^{\frac{1}{2}}(\partial\Omega)$ and $n(x)$ is the exterior normal defined on $\partial\Omega$.
So before going on, we need some hypotheses , let us assume that:

(i) - Ω is a $\mathcal{C}^2-$ regular open domain.

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(ii) -$(\frac{\partial u}{\partial n}) = c > 0$ (a positive constant).

Proposition 2.2.1 *(Hard obstacle case)*
Suppose that Ω is a critical point, then

$$\begin{aligned} Q(v) &= d^2\lambda(\Omega; V; V) \\ &= d^2 J(\Omega; V; V) \\ &= -2\beta_\Omega \int_{\partial K} (N-1)Hv^2 ds - 2\beta_\Omega \int_\Omega |\nabla \Lambda|^2 dx \\ &= -2\beta_\Omega \int_{\partial K} (N-1)Hv^2 ds - 2\beta_\Omega \int_{\partial K} vLvds \end{aligned}$$

Where β_Ω is the Lagrange multiplier, here it is negative and Λ is the solution of the following boundary value problem

$$\begin{cases} -\Delta\Lambda = 0 & in \quad D\backslash K \\ \Lambda = v & on \quad \partial K \\ \Lambda = 0 & on \quad \partial D. \end{cases} \tag{2.7}$$

H is the mean curvature of ∂K and L is a pseudo differential operator known as the Steklov-Poincaré or capacity or Dirichlet to Neumann(see e.g [16]) operator, defined by $Lv = \frac{\partial \Lambda}{\partial n}$ and n is the unit exterior normal of K. In fact Λ is the harmonic extension of v in Ω.

Proof of the Proposition (3.3.1)
We use the definition of the derivative with respect to the domain and we apply it to $\lambda'(\Omega, V)$. Then we get

$$\begin{aligned} Q(v) &= d^2\lambda(\Omega, V, V) \\ &= \int_{\Omega\backslash K} (\mathrm{div}((-|\nabla u|^2)V(x,0)))' dx + \int_{\Omega\backslash K} \mathrm{div}(V(x,0)\mathrm{div}(-|\nabla u|^2)V(x,0))dx \\ Q(v) &= -[\int_{\partial K} 2\nabla u \nabla u' V(x,0).n + \mathrm{div}((|\nabla u|^2)V(x,0)V(x,0).n]ds. \end{aligned}$$

As

$$-\frac{\partial u}{\partial n} = c \;\; a.e \;\; \text{on} \;\; \partial K.$$

Recall, $u = 0$ on ∂K, and : $\nabla u = \frac{\partial u}{\partial n} n = -cn$. So

$$Q(v) = -[\int_{\partial K} -2cn.\nabla u' V(x,0).n + \mathrm{div}(|\nabla u|^2 V(x,0))V(x,0).n]ds.$$

We have $u^{'} = -\dfrac{\partial u}{\partial n}V.n = cV.n$ on ∂K. And since $-\dfrac{\partial u}{\partial n} = c$ $a.e$ on ∂K and $V.n = v$, we get $u^{'} = cv$ on ∂K and $n.\nabla u^{'} = \dfrac{\partial u^{'}}{\partial n} = c\dfrac{\partial v}{\partial n} = cLv,$ where L is a pseudo differential operator, defined by $Lv = \dfrac{\partial \Lambda}{\partial n}$ such that

$$\begin{cases} -\Delta\Lambda & = & 0 & \text{in} & \Omega\backslash K \\ \Lambda & = & 0 & \text{on} & \partial K \\ \Lambda & = & v & \text{on} & \partial\Omega, \end{cases} \tag{2.8}$$

Λ is the extension of v in $\Omega\backslash K$.
Thus

$$Q(v) = \int_{\partial K}(2c^2 vLv - \text{div}((|\nabla u|^2)vn)v)ds$$

Note that

$$\text{div}(|\nabla u|^2 vn) = v\nabla(|\nabla u|^2).n = 2v|\nabla u|\nabla(|\nabla u|).n$$

Sine we assumed that Ω is $\mathcal{C}^2$,so using the formula of the level motion set related to the mean curvature.
In fact $\partial K = \{x \in \mathbb{R}^N;\ u(x) = 0\}$ and we have

$$-(N-1)H = \text{div}(\frac{\nabla u}{|\nabla u|}) = \frac{\Delta u}{|\nabla u|} - \frac{\nabla u.\nabla(|\nabla u|)}{|\nabla u|^2}.$$

where H is the mean curvature of $\partial\Omega$. Furthermore, since $u = 0$ on ∂K, we have $\Delta u = 0$ on ∂K.
Finally we get

$$\begin{aligned} (N-1)H &= \frac{-cn}{|\nabla u|^2}.\nabla(|\nabla u|) \quad \text{i.e} \\ -(N-1)H|\nabla u|^2 &= cn.\nabla(|\nabla u|) \\ c\nabla(|\nabla u|^2) &= -(N-1)H|\nabla u|^2 n \\ \nabla(|\nabla u|^2).n &= \frac{-(N-1)}{c}H|\nabla u|^2 \quad \text{,hence} \\ \text{div}(|\nabla u|^2 vn) &= v(\frac{-2(N-1)}{c}H)|\nabla u|^3 \quad \text{then} \\ Q(v) &= \int_{\partial K}(2c^2 vLv + 2c^2(N-1)Hv^2 ds \\ &= \int_{\partial K}(-2\beta_\Omega vLv - 2\beta_\Omega(N-1)Hv^2 ds \end{aligned}$$

And by the Green's formula we get

$$\int_{\partial K} vLvds = \int_{\Omega\setminus K} |\nabla\Lambda|^2 dx.$$

■

Remark 2.2.1 *Let us note that since the Lagrange multiplier β_Ω is negative, we have to take into account this information in the search for local strict minimum. In fact $Q_1 = -\beta_\Omega Q(v)$ where $Q(v)$ is the quadratic form computed in the free boundary part of this paper.*
For the quadratic form Q_1 to be positive, we only need Q to be positive, on cane conclude easily as follows:
if $Q > 0$ then $Q_1 > 0$.
sufficient conditions for the strict local minimum are the same as in the case of free boundary problem obtained in [2].

Proposition 2.2.2 *(Well case)*
Let's suppose that Ω is a critical point , then for any $v \in H^{\frac{1}{2}}(\partial\Omega)$, we have

$$\begin{aligned} Q(v) &= d^2\lambda(\Omega; V; V) \\ &= 2\alpha \int_{\partial K} (v^2 u \frac{\partial u}{\partial n} + u'uv) d\sigma = 0 \end{aligned}$$

Proof of the Proposition (2.2.2)
Using the same techniques in the previous proof (3.3.1)we get the demonstration.

Remark 2.2.1 *In fact $u' = \frac{\partial u}{\partial n}\ v$ on ∂K.*
No information can be obtained from only the calculation of the second derivative. We have a degenerate situation. We think that it would be a good challenge to study this situation.

2.3 Sufficient conditions for the minimum

To give sufficient conditions for a local minimum of basic worth, we first present the results we obtained in our paper [2].
Let A be an operator defined in the following sense :

$$\begin{aligned} A : H^{\frac{1}{2}}(\partial\Omega) &\longrightarrow H^{-\frac{1}{2}}(\partial\Omega) \\ A &= L + (N-1)(||H^-||_\infty + H)I, \end{aligned}$$

where I is the identity operator? L is the pseudo differential operator as defined in the proposition (3.3.1), and $H^- = max(0, -H)$.

Remark 2.3.1
As assumed $\partial\Omega$ is of class $\mathcal{C}^2$, then the mean curvature H is a continuous function on $\partial\Omega$.
Let us set $\alpha(x) = (N-1)(||H^-||_\infty + H(x)),\ \forall\ x\ \in\ \partial\Omega$. We note that α is continous and $\forall\ x\ \in\ \partial\Omega,\ \alpha(x)\ \geq\ 0$ (moreover $\alpha(x) > 0$ on a sufficiently large set).

Lemma 2.3.1

1 - The operator A is a bijection from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$ and it is continous.

2 - The inverse operator A^{-1} is compact and self adjoint from $H^{-\frac{1}{2}}(\partial\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$.

Proof
For proof see [2]

Remark 2.3.2 *Since the inverse operator : $(\alpha I + L)^{-1}$ is compact, self adjoint, then there exists a Hilbert basis $(\phi_n)_{(n\ \in\ \mathbb{N})}\ \subset\ H^{\frac{1}{2}}(\partial\Omega)$ and a decreasing sequence of eigenvalues μ_n which goes to 0.*

Proposition 2.3.1
Let Ω_0 the critical shape for λ_{Ω_t} (The first eigenvalue of the Laplace operator with Dirichlet boundary conditions).is given by

$$\lambda_{\Omega_t} = \min_{u\in W_0^{1,2}(\Omega),u\neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

With Ω_t solution of the following problem:

$$\begin{cases} -\Delta u_\Omega & = & \lambda_\Omega u_\Omega & in & \Omega_t \\ u & = & 0 & on & \partial\Omega_t \end{cases} \tag{2.9}$$

Ω_0 is a local strict minimum of λ_{Ω_t} if and only if

$$(N-1)||H^-||_\infty \ < \ \frac{1}{\mu_0}$$

Proof
Since $v\ \in\ H^{\frac{1}{2}}(\partial\Omega)$,then

$$\begin{aligned} v &= \sum_{n=0}^{\infty} v_n\phi_n \\ (L + (N-1)(H + ||H^-||_\infty)I)^{-1}\phi_n &= \mu_n\phi_n \text{ then} \\ (L + (N-1)H)I)\phi_n &= (\frac{1}{\mu_n} - (N-1)||H^-||_\infty)\phi_n \end{aligned}$$

Let us set $\lambda_n = \dfrac{1}{\mu_n} - (N-1)||H^-||_\infty$.
So (λ_n) is an increasing sequence going to infinity. Then we have

$$\frac{Q(v)}{-2\beta_\Omega} = < (L + (N-1)HI)v, v > \quad (2.10)$$

$$= \sum_{n=0}^{\infty} \lambda_n |v_n|^2. \quad (2.11)$$

Suppose that

$$(N-1)||H^-||_\infty < \frac{1}{\mu_0}$$

So

$$\lambda_0 = \frac{1}{\mu_0} - (N-1)||H^-||_\infty > 0$$

As $\lambda_0 > 0$ then as a result of λ_n increasing we get

$$Q(v) \geq c^2 \lambda_0 ||v||^2_{H^{\frac{1}{2}}(\partial\Omega)}$$

with $c^2 = -2\beta_\Omega$
We are now in the case where it is possible to use the main result in [13],which can be formulated as follows: there is a positive constant C and a positive function w and $\lim_{\eta \longrightarrow 0} w(\eta) = 0$ such that

$$|j''(t) - j''(0)| \leq C w(\eta) \quad (2.12)$$

where $j''(t) = d^2\lambda(\Omega_t, V, V)$, $j(t) = \lambda(\Omega_t)$ and $j''(0) = d^2\lambda(\Omega, V, V) = Q(v)$, $j(0) = \lambda(\Omega)$.
We replace in (2.12) we give

$$|d^2\lambda(\Omega_t, V, V) - d^2\lambda(\Omega, V, V)| \leq C w(\eta) \quad (2.13)$$

Using Taylor formula with integral residual we get: $\lambda(\Omega_t) = \lambda(\Omega) + \int_0^1 (1-t) d^2\lambda(\Omega_t, V, V) dt$.
Using inequality (2.13) :
if $\lambda_0 > 0$, Ω is a local strict minimum for the functional λ_Ω.
If Ω is a convex domain, then $\lambda_0 = \dfrac{1}{\mu_0} > 0$. Hence Ω is a local strict minimum for the functional λ_Ω.

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conclusion

Let Ω be a critical form of the first eigenvalue of the Laplace operator with Dirichlet boundary conditions,
from proposition (3.4.1), we conclude that:

- if $(N-1)||H^-||_\infty < \dfrac{1}{\mu_0}$, Ω is a local strict minimum for the first eigenvalue λ_Ω of the Laplace operator .
- if Ω is a convex domain, then $\dfrac{1}{\mu_0} > 0$. Hence Ω is a local strict minimum for the first eigenvalue λ_Ω of the Laplace operator .

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Chapter 3

The problem of obstacle for the first eigenvalue for the p-laplacian operator

In [32] Long-Jiang Gua, Xiaoyu Zengb, and Huan-Song Zhoub have studied the existence of asymptotic behavior of the base states for the eigenvalue problem of the following p-laplacian equation:

$$\Delta_p u = V(x)|u|^{p-2}u = \mu|u|^{p-2}u + a|u|^{s-2}u, \ \ x \in \mathbb{R}^N$$

with $p \in (1, n)$, $s = p + \frac{p^2}{n}, a \geqslant 0$ and $\mu \in \mathbb{R}$ Is a parameter and $V(x)$ Is a field of vectors satisfying certain assumptions.
In [31] Leandro , Del Pezzo and Julio Studied the first eigenvalue for the p-Laplacian operator with the boundary conditions of Dirichlet and Neumann (mixed boundary conditions). They considered the following problem:

$$\begin{cases} \Delta_p u = & \lambda\alpha|u|^{\alpha-2}u|v|^{\beta} \quad \text{on} \quad \Omega \\ \Delta_q u = & \lambda\beta|u|^{\alpha}u|v|^{\beta-2}v \quad \text{on} \quad \Omega \end{cases} \tag{3.1}$$

with $\frac{\alpha}{p} + \frac{\beta}{q} = 1$ and Next mixed boundary conditions:

$$u = 0, \ \ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} \ \text{sur} \ \partial\Omega$$

In [33] idrissa ly studied the behavior of the first eigenvalue of the p-Laplacian operator $\lambda_1^p(\Omega_n)$ avec la condition du Dirichlet homogène au bord du domaine variable (Ω_n) ,où (Ω_n) est une famille séquentielles des perturbations géométrie.
In [14] Daniele Valtorta gave the estimate of the first non-trivial eigenvalue of the p-Laplacian on a compact Riemannian manifold with a non-negative Ricci curvature

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and characterize the case of equality. He studied the following problem:

$$\begin{cases} \Delta_p(u) &= \lambda_{1,p}|u|^{\alpha-2}u & \text{on} \quad \Omega \\ \langle \nabla u \; n \rangle &= 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.2}$$

Daniele Valtorta has proved the following strong estimate:

$$\frac{\lambda_{1,p}}{p-1} \geqslant \frac{\Pi_p^p}{d^p}$$

With

$$\Pi_p = \int_{-1}^{1} \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin(\frac{\pi}{p})}$$

In this article we will study the obstacle position problem for the p-Laplacian operator.
The obstacle locating problem for the fundamental eigenvalue is to locate the position of the obstacle placement so as to maximize or minimize the eigenvalue of the p-Laplacian operator. We were interested in the following problems:
Let Ω be a bounded open set of $\mathbb{R}^N$ and K An obstacle that moves The interior of D. We consider the problem :

$$\begin{cases} \Delta_p(u) &= \lambda_{1,p}|u|^{\alpha-2}u & \text{in} \quad D\backslash K \\ u &= 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.3}$$

Let $\lambda_{1,p}$ The first eigenvalue of the p-Laplacian operator with certain hypotheses we want to give the necessary and sufficient conditions so that the first eigenvalue of the p-Laplacian operator is minimal or we want to determine the position of K in Ω so that $\lambda_{1,p}$ Is minimal where $\lambda_{1,p}$ Represents the first eigenvalue of the p-Laplacian operator

3.1 Position of the problem

Let D A fixed open set of $\mathbb{R}^N$ and K An obstacle that is a subset of D. In this work we study the minimization of the first eigenvalue of the operator P-Laplacian with conditions at the edges of Dirichlet null on the border of $\Omega = D/K$. . More specifically,, we place K at The interior of D With conditions at the edges of Dirichlet null on the border of $\Omega = D/K$.
The question is: : We want to determine the position of K in D so that $\lambda_{1,p}$ minimal where $\lambda_{1,p}$ Represents the first eigenvalue of the p-Laplacian Dirichlet

$$\begin{cases} \Delta_p(u) &= \lambda_{1,p}|u|^{\alpha-2}u & \text{in} \quad \Omega \\ u &= 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.4}$$

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With the p-laplacian operator defined by the following relation:

$$\Delta_p : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega)$$
$$\Delta_p \ u \longrightarrow div(|\nabla u|^{P-2}\nabla u)$$
$$W^{-1,q}(\Omega) \text{ la dual de } W_0^{1,p}(\Omega)$$

Define a vector field

$$\begin{array}{ll} V:\mathbb{R}^N & \longmapsto \mathbb{R}^N \\ x & \longmapsto (V_1(x),\ V_2(x),\ V_3(x)...,V_N(x)) \end{array}$$

For all real t Small, we define the domains Disturbed:

$$\Omega_t = (Id+tV)(\Omega) = \{x+tV(x), \quad x \in \Omega\}.$$

The variation of Ω Is explained by the fact that K Moves into Ω Without going out. if K Is a hard obstacle, the movement of K dans Ω Is done either by translation or by rotation, or one combines these two Types of motion.
If K Is considered a soft obstacle, K May undergo a transformation by homothety.
After perturbation of the problem (3.4) becomes :

$$\left\{ \begin{array}{rcll} \Delta_p(u_t) & = & \lambda_{1,p}|u_t|^{\alpha-2}u_t & \text{in} \quad \Omega_t \\ u_t & = & 0 & \text{on} \quad \partial\Omega_t \end{array} \right. \tag{3.5}$$

By using the variational formulation the first eigenvalue of the operator P-Laplacian is defined by the following Rayleigh nonlinear quotient

$$\lambda_1(\Omega_t) = \min_{u\in W_0^{1,p}(\Omega),u\neq 0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

$W_0^{1,p}$ The adherence of all functions $\mathbf{C}^\infty$ Has compact media contained in $W^{1,p}$, with $W^{1,p}(\Omega) = \{u \in L^p(\Omega); \frac{\partial u}{\partial x_i} \in L^p(\Omega), (i=1,....,N)\}$
We will give some definitions before we formulate more precisely the problem

Definition 3.1.1 *Let ξ A unit vector of $\mathbb{R}^N$, ε A strictly real number Positive and y belonging to $\mathbb{R}^N$, The summit cone y And direction ξ, Of angle at the top and height ε. The set defined by*

$$C(y,\xi,\varepsilon,\varepsilon) = \{x \in \mathbb{R}^N : |x-y| \leq \varepsilon \ et \ |(x-y) \cdot \xi| \geq |x-y| cos \ \varepsilon\}$$

Definition 3.1.2 *Let Ω An open set of $\mathbb{R}^N$,
Ω Has the property of ε - Cone if for any
$x \in \partial\Omega$, There is a direction ξ And a strictly positive number ε such as*

$$C(y,\xi,\varepsilon,\varepsilon) \subset \Omega \quad pour\ tout\ y \in B(x,\varepsilon) \cap \overline{\Omega}$$

In the case of a hard obstacle:
Define for any real t pretty small $T_t(B)$ Such as a translation, rotation, or face.
Let $J_2(\Omega_t) = \int_{\Omega_t} dx - v_o$ with $v_o > 0$ and
$\Theta_\varepsilon = \Big\{\Omega_t = D\backslash T_t(B),$ open of $\mathbb{R}^N$ And verifying ownership of the ε
Cone and $\int_{\Omega_t} dx = v_o\Big\}$
So the problem becomes: determine the position of B such than

$$\min_{\Omega_1\in\Theta_\varepsilon} \lambda_1, p(\Omega_t) \text{ Is reached}$$

$$\text{or } \lambda_1, p(\Omega_t) = \min_{u\in W^{1,p}(\Omega)}\Big\{\int_{\Omega_t} |\nabla u|^p dx \ : \ \int_{\Omega_t} |u|^p dx = 1,\Big\}$$

3.2 The shape Critical of the first eigenvalue for the p-Laplace operator $(\lambda_{1,p}(\Omega_t))$

Now we will start by studying the shape critical of the functional $\lambda_1, p(\Omega_t)$ (the first eigenvalue for the p-laplacian operator) above. Indeed, the solution domain of the free boundary problem is not automatically a minimum for the function $\lambda_{1,p}(\Omega_t)$. this justifies the study of the shape critical of $\lambda_{1,p}(\Omega_t)$, followed by l The study of the quadratic form. The functional $\lambda_{1,p}(\Omega_t)$ is given by the following relation:

$$\lambda_{1,p}(\Omega_t) = \int_{\Omega_t} |\nabla u|^p dx$$

With u solution of the following problem:

$$\begin{cases} \Delta_p(u_t) &= \lambda_{1,p}|u_t|^{\alpha-2}u_t \quad \text{in} \quad \Omega_t \\ u_t &= 0 \qquad\qquad\qquad \text{on} \quad \partial\Omega_t \end{cases} \tag{3.6}$$

Using the hadamard formula we get

$$\lambda'_{1,p}(\Omega_t) = \int_{\Omega_t} (|\nabla u|^p)' dx + \int_{\Omega_t} div(|\nabla u|^p V(0))dx$$

$$\lambda'_{1,p}(\Omega_t) = \int_{\Omega_t} \nabla u \nabla u' |\nabla u|^{P-2} dx + \int_{\partial K} |\nabla u|^p V(0).ndx$$

With n is the unit exterior normal of K
with u' Is the derivative of the form of u , and u' Satisfying the following equation:

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$$\left\{\begin{array}{rcl} -div(\nabla u'|\nabla u|^{p-2})-(p-2)div(|\nabla u|^{p-4}(\nabla u.\nabla u')\nabla u) & = & \lambda_{1,p}'|u_t|^{\alpha-2}u \\ +(p-1)\lambda_{1,p}|u_t|^{p-2}u' & \text{in} & \Omega\backslash K \\ u' & = & -\frac{\partial u}{\partial n}V.n \qquad \text{on} \quad \partial\Omega_t \end{array}\right. \tag{3.7}$$

We multiply the equation (3.7) by u and using Green formula we get

$$-(p-1)\int_{\Omega_t}-div(\nabla u'|\nabla u|^{p-2})udx=\lambda_{1,p}'(\Omega,V)+\int_{\Omega_t}(p-1)\lambda_{1,p}|u_t|^{p-2}u'udx$$

Using the Green's formula then we get

$$-(p-1)\int_{\Omega_t}-div(\nabla u|\nabla u|^{p-2})u'dx+-(p-1)\int_{\partial K}|\nabla u|^{p-2}\nabla u.nu'dx=$$
$$\lambda_{1,p}'(\Omega,V)+\int_{\Omega_t}(p-1)\lambda_{1,p}|u_t|^{p-2}u'udx \tag{3.8}$$

We are getting

$$-(p-1)\int_{\Omega_t}(-div(\nabla u|\nabla u|^{p-2})-\lambda_{1,p}|u_t|^{p-2}u)u'dx+-(p-1)\int_{\partial K}|\nabla u|^{p-2}\nabla u.nu'dx=\lambda_{1,p}'(\Omega,V) \tag{3.9}$$

What gives after the simplification

$$\lambda_{1,p}'(\Omega,V)=-(p-1)\int_{\partial K}|\nabla u|^{p-2}\nabla u.nu'd\sigma$$

since

$$u'=-\frac{\partial u}{\partial n}V.n= \quad \text{on} \quad \partial K$$

and

$$\nabla u.n=\frac{\partial u}{\partial n}=-|\nabla u|$$

So we get

$$\lambda_{1,p}'(\Omega,V)=-(p-1)\int_{\partial K}|\nabla u|^{p}V(0).nd\sigma$$

Since we have an optimization problem with equality constraint

$$J_2(\Omega)=\int_{\Omega}dx-v_o=0$$

So there is a Lagrange multiplier $\beta < 0$ Depending on the domain Ω_t and verifying

$$\lambda'_{(1,p)}(\Omega, V) = \beta dJ_2(\Omega, V) \tag{3.10}$$

The derivative of $J_2(\Omega, V)$ Is given by

$$dJ_2(\Omega, V) = \int_{\partial K} V \cdot n \, d\sigma \tag{3.11}$$

By replacing in (3.10)

$$-(p-1)\int_{\partial K} |\nabla u|^p V(0) \cdot n d\sigma = \beta \int_{\partial K} V(0) \cdot n \, d\sigma \tag{3.12}$$

Which give

$$-(p-1)|\nabla u|^p = \beta \quad \text{on} \quad \partial K \tag{3.13}$$

So we get the following relation:

$$|\nabla u| = (\frac{-\beta}{p-1})^{\frac{1}{P}} \quad \text{on} \quad \partial K \tag{3.14}$$

Let $\Omega_t = \Omega \backslash K$ with K An obstacle that moves inside of Ω So $\Omega_0 = \Omega \backslash K$ Is a shape critical of functional $\lambda_{1,p}(\Omega)$ If and if there exists a multiplier of lagrange $\beta < 0$ Depends on domain Ω_t Verifying the following relation:

$$|\nabla u| = (\frac{-\beta}{p-1})^{\frac{1}{P}} \quad \text{on} \quad \partial K \tag{3.15}$$

3.3 Quadratic form associated with the first eigenvalue of the p-Laplace operator $(\lambda_{1,p}(\Omega_t))$

We have just proved that $\Omega_t = \Omega \backslash K$ Is a shape critical of the functional $\lambda_{1,p}(\Omega)$. And our goal is to know if Ω_t can be the minimum of $\lambda_{1,p}(\Omega)$ under certain assumptions. This leads us to the study of the positivity of a quadratic form that we will denote by Q. This quadratic form is obtained by calculating the second derivative of $\lambda_{1,p}(\Omega)$ with respect to the domain. So before we go on, we need some assumptions. Suppose that:

(i) Ω Is open of class $\mathcal{C}^2-$ regular.

(ii) $V(x;t) = v(x)n(x), \; v \in H^{\frac{1}{2}}(\partial\Omega), \; \forall \, t$ in $[0, \epsilon[$.

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Proposition 3.3.1 *(Hard obstacle case)*
Suppose that Ω_0 is a shape critical, then quadratic form associated with the first eigenvalue of the p-Laplace operator is given by:

$$\begin{aligned} Q(v) &= d^2\lambda_{(1,p)}(\Omega;V;V) \\ &= -p\beta_{\Omega_t}\int_{\partial K}(N-1)Hv^2ds - p\beta_{\Omega_t}\int_{\Omega_t}|\nabla\Lambda|^2dx \\ &= -p\beta_{\Omega_t}\int_{\partial K}vLvd\sigma - p\beta_{\Omega_t}(N-1)\int_{\partial K}Hv^2d\sigma \end{aligned}$$

Where β_Ω is the Lagrange multiplier ,here it is negative and p is allowed to range over $1 < p < \infty$, and Λ is the solution of the following boundary value problem

$$\begin{cases} -\Delta\Lambda = 0 & in \quad \Omega_t = D/K \\ \Lambda = 0 & on \quad \partial D \\ \Lambda = v & on \quad \partial K \end{cases} \tag{3.16}$$

H is the mean curvature of ∂K and L is a pseudo differential operator known as the Steklov-Poincaré or capacity or Dirichlet to Neumann(see e.g [16]) operator, defined by $Lv = \dfrac{\partial\Lambda}{\partial n}$ and n is the unit exterior normal of K. In fact Λ is the harmonic extension of v in Ω.

Proof
The first derivative of the functional $\lambda_{1,p}(\Omega)$ Is given by the following equation

$$\lambda'_{1,p}(\Omega,V) = -(p-1)\int_{\partial K}|\nabla u|^pV(0).nd\sigma$$

$$\lambda'_{1,p}(\Omega,V) = -(p-1)\int_{\Omega_t}div(|\nabla u|^pV(0))dx$$

Using the hadamard formula we get

$$d^2\lambda_{1,p}(\Omega,V,V) = -(p-1)\int_{\Omega_t}div(|\nabla u|^pV(0))'dx + -(p-1)\int_{\Omega_t}div(div(|\nabla u|^pV(0))V(0))dx$$

What gives after the simplification

$$\frac{-1}{(p-1)}d^2\lambda_{1,p}(\Omega,V,V) = \int_{\Omega_t}div(p|\nabla u|^{p-2}\nabla u\nabla u'V(0))dx + \int_{\Omega_t}div(div(|\nabla u|^pV(0))V(0))dx$$

$$\frac{-1}{(p-1)}d^2\lambda_{1,p}(\Omega,V,V) = \int_{\partial K} p|\nabla u|^{p-2}\nabla u \nabla u' V(0).nd\sigma + \int_{\partial K} div(|\nabla u|^p V(0))V(0).nd\sigma$$

$$\frac{-1}{(p-1)}d^2\lambda_{1,p}(\Omega,V,V) = \int_{\partial K} p|\nabla u|^{p-2}\nabla u \nabla u' V(0).nd\sigma + \int_{\partial K} \nabla(|\nabla u|^p)V(0)V(0).nd\sigma$$

Since $|\nabla u| = (\frac{-\beta}{p-1})^{\frac{1}{P}}$ on ∂K and $\nabla u = -|\nabla u|.n$ on ∂K

$$\text{and } u' = -\frac{\partial u}{\partial n}V.n = |\nabla u|v \text{ on } \partial K$$

What gives after the simplification

$$\frac{d^2\lambda_{1,p}(\Omega,V,V)}{(p-1)} = \int_{\partial K} [p|\nabla u|^{p-1}n.\nabla u' V(0).n - \nabla(|\nabla u|^p)V(0)V(0).n]d\sigma \quad (3.17)$$

Since $u' = -\frac{\partial u}{\partial n}V.n = (\frac{-\beta}{p-1})^{\frac{1}{P}}v$ on ∂K So $n.\nabla u' = (\frac{-\beta}{p-1})^{\frac{1}{P}}\frac{\partial v}{\partial n}$ on ∂K
Replacing in (3.17) We obtain the following relation :

$$\frac{d^2\lambda_{1,p}(\Omega,V,V)}{(p-1)} = \int_{\partial K} [p(\frac{-\beta}{p-1})\frac{\partial v}{\partial n}v - \nabla(|\nabla u|^p)V(0)V(0).n]d\sigma \quad (3.18)$$

Replacing in (3.18) Which give:

$$\frac{d^2\lambda_{1,p}(\Omega,V,V)}{(p-1)} = p(\frac{-\beta}{p-1})\int_{\partial K} vLvd\sigma - \int_{\partial K} v^2\nabla(|\nabla u|^p).nd\sigma \quad (3.19)$$

So we get the following relation:

$$d^2\lambda_{1,p}(\Omega,V,V) = -p\beta\int_{\partial K} vLvd\sigma - (p-1)\int_{\partial K} v^2\nabla(|\nabla u|^p).nd\sigma \quad (3.20)$$

Sine we assumed that Ω is $\mathcal{C}^2$,so using the formula of the level motion set related to the mean curvature.
In fact $\partial K = \{x \in \mathbb{R}^N;\ u(x) = 0\}$ and we have

$$-(N-1)H = \text{div}(\frac{\nabla u}{|\nabla u|}) = \frac{\Delta u}{|\nabla u|} - \frac{\nabla u.\nabla(|\nabla u|)}{|\nabla u|^2}.$$

where H is the mean curvature of ∂K. Furthermore, since $u = 0$ on ∂K, we have $\Delta_p u = 0$ on ∂K.

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Finally we get

$$\begin{aligned}(N-1)H &= \frac{-|\nabla u|n}{|\nabla u|^2}.\nabla(|\nabla u|) \quad \text{i.e} \\ -(N-1)H|\nabla u| &= n.\nabla(|\nabla u|) \\ \nabla(|\nabla u|) &= -(N-1)H|\nabla u|.n\end{aligned}$$

By miltipliating by $|\nabla u|^{P-1}$ we are getting:

$$|\nabla u|^{P-1}\nabla(|\nabla u|) = -(N-1)H|\nabla u|^p.n$$

Finally we get

$$\nabla(|\nabla u|^p) = -p(N-1)H|\nabla u|^p.n$$

This gives the following relation:

$$\nabla(|\nabla u|^p).n = \frac{Hp(N-1)\beta}{(p-1)}$$

Replacing in (3.20) We obtain the following relation :

$$d^2\lambda_{1,p}(\Omega, V, V) = -p\beta\int_{\partial K} vLvd\sigma - p\beta(N-1)\int_{\partial K} Hv^2 d\sigma \tag{3.21}$$

Therefore the quadratic form of the functional $\lambda_{1,p}(\Omega_t)$ Is given by the following equation:

$$\begin{aligned}Q(v) &= d^2\lambda_{(1,p)}(\Omega; V; V) \\ &= -p\beta_{\Omega_t}\int_{\partial K}(N-1)Hv^2 ds - p\beta_{\Omega_t}\int_{\Omega_t}|\nabla\Lambda|^2 dx \\ &= -p\beta_{\Omega_t}\int_{\partial K} vLvd\sigma - p\beta_{\Omega_t}(N-1)\int_{\partial K} Hv^2 d\sigma\end{aligned}$$

3.4 Sufficient conditions for the minimum of the first eigenvalue of the p-Laplace operator

In [13], Michel Pierre and Marc Dambrine ((See as well [10],[11]) Have shown that it is not enough to prove that the quadratic form is positive to say that a critical form is a minimum.
For $t \in [0, \epsilon[$, $\lambda_{(1,p)}(\Omega_t) = \lambda_{(1,p)}(\Omega) + \lambda'_{(1,p)}(\Omega_t, V)t + d^2\lambda_{(1,p)}(\Omega_t; V; V)t^2 + o(t^2)$.

The amount $o(t^2)$ Is expressed in terms of the norm of $\mathcal{C}^2$. It appears in the expression $d^2J(\Omega, V, V)$ norm of $H^{\frac{1}{2}}(\partial\Omega)$. And these two norm are not equivalent. The amount $o(t^2)$ Is not lower than $||V||^2_{H^{\frac{1}{2}}(\partial\Omega)}$ see [13],[19].
next, such an argument does not guarantee that the critical point is A strict local minimum. For this we will use the main result [13] and The Taylor formula with With integral rest to see if Ω is a strict local minimum or not.
To give sufficient conditions for a local minimum of basic worth, we first present the results we obtained in our paper [2].
Let A be an operator defined in the following sense :

$$\begin{array}{rcl} A : H^{\frac{1}{2}}(\partial\Omega) & \longrightarrow & H^{-\frac{1}{2}}(\partial\Omega) \\ A & = & L + (N-1)(||H^-||_\infty + H)I, \end{array}$$

where I is the identity operator L is the pseudo differential operator as defined in the proposition (3.3.1), and $H^- = max(0, -H)$.

Remark 3.4.1
As assumed $\partial\Omega$ is of class $\mathcal{C}^2$, then the mean curvature H is a continuous function on ∂K.
Let us set $\alpha(x) = (N-1)(||H^-||_\infty + H(x)),\ \forall\ x\ \in\ \partial\Omega$. We note that α is continous and $\forall\ x\ \in\ \partial\Omega,\ \alpha(x)\ \geq\ 0$ (moreover $\alpha(x) > 0$ on a sufficiently large set).

Lemma 3.4.1

1 - The operator A is a bijection from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$ and it is continous.

2 - The inverse operator A^{-1} is compact and self adjoint from $H^{-\frac{1}{2}}(\partial\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$.

Proof
For proof see [2]

Remark 3.4.2 *Since the inverse operator : $(\alpha I + L)^{-1}$ is compact, self adjoint, then there exists a Hilbert basis $(\phi_n)_{(n\,\in\,\mathbb{N})}\ \subset\ H^{\frac{1}{2}}(\partial\Omega)$ and a decreasing sequence of eigenvalues μ_n which goes to 0.*

Proposition 3.4.1
Let Ω_0 the critical shape for $\lambda_{1,p}(\Omega_t)$ (The first eigenvalue of the p-Laplace operator) is given by

$$\lambda_{1,p}(\Omega_t) = \min_{u\in W_0^{1,p}(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

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With Ω_t solution of the following problem:

$$\begin{cases} \Delta_p(u_t) = \lambda_{1,p}|u_t|^{\alpha-2}u_t & in \quad \Omega_t \\ u_t = 0 & on \quad \partial\Omega_t \end{cases} \tag{3.22}$$

Ω_0 is a local strict minimum of $\lambda_{1,p}(\Omega_t)$ if and only if

$$(N-1)||H^-||_\infty \ < \ \frac{1}{\mu_0}$$

Proof
The proof is a direct consequence of the remarks(3.4.1) ,(3.4.2) and The results of [?]

conclusion

Let $\Omega_t = \Omega\backslash K$ with K An obstacle that moves Inside of Ω So $\Omega_0 = \Omega\backslash K$ Is a critical shape of the functional $\lambda_{1,p}(\Omega)$ If and if there is a lagrange multiplier $\beta < 0$ Depends on the domain Ω_0 verifying the following relation:

$$|\nabla u| = (\frac{-\beta}{p-1})^{\frac{1}{P}} \quad \text{on} \quad \partial K \tag{3.23}$$

Let Ω_0 be a critical form of the first eigenvalue $(\lambda_{1,p}(\Omega))$ of the P-Laplace operator with Dirichlet boundary conditions, from proposition (3.4.1), we conclude that:
if $(N-1)||H^-||_\infty \ < \ \frac{1}{\mu_0}$, Ω is a local strict minimum for the first eigenvalue $(\lambda_{1,p}(\Omega))$ of the P-Laplace operator .

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