Matching in Semigraph

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Abstract

In this paper, the well known concept of matching of graph theory has been discussed in the setting of semigraphs resulting in few new concepts like maximal vertex-saturated matching, minimal edge-saturated matching and optimum matching which have no parallels in graphs. In this connection, we record a number of characterizations of maximum matching and other related terms of semigraphs developed here and also establish parallels of theorems due to Berge and König for graphs. A new concept on total adjacent domination in connection with adjacent domination in semigraph as studied by Kamath and Bhat [7] is introduced here and few results are developed that establish links between adjacent domination number and total adjacent domination number as well as between total adjacent domination number and minimal edge-saturated matching.

Keywords: - Semigraph, Total adjacent domination, Matching, Maximum matching, Maximal vertex-saturated matching, Minimal edge-saturated matching, Optimum matching.

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1. Introduction

The matching is one of the most interesting and well-studied concept in Graph Theory and it has vast applications in real world situations. One of the most significant results about this concept is due to C. Berge [4], which gives the characterization of a maximum matching. D. König [6] and P. Hall [5] established results on matching for a bipartite graph that assume much significance in graph theory.

Coming to the context of this paper it may be mentioned that there are two important generalizations of graphs, one of which is called a hypergraph, while another is called a semigraph. The former being a generalization by C. Berge [4] the latter is due to E. Sampathkumar [2].While both hypergraph and semigraph allow edges with more than two vertices, however, the vertices in any edge of a semigraph follow a particular order though the vertices in a hypergraph have no such order. The different modifications in edge structure of both the generalizations have widened the periphery of applications and interpretations of graphs to real situations in different perspectives. So far the semigraphs are concerned this fact has been justified by simple examples here.

The pioneering works on “Semigraphs” by E. Sampathkumar [2] have already created much interest and enthusiasm among the graph theorists and the flow of development of this newly born idea is on the rising trend. In [7], S. S. Kamath and R. S. Bhat introduced three types of domination in semigraphs. Y. B. Venkatakrishnan and V. Swaminathan [8] introduced the domination and independence parameters for the bipartite semigraph. X_a-chromatic number, X_a-hyperindependent number and X_a-irredundant number are few other concepts in semigraph defined by them [8].

2. Preliminaries

The terminology and notations used here are as in [1] and [2] unless otherwise specified.

A **Semigraph** *G* is an ordered pair (*V*,*X*), where *V* is a non-empty set of points, called vertices of *G* and *X* is a set of *n*-tuples ($n\geq 2$) of distinct vertices of G, called edges satisfying the following conditions:

(a) Any two edges have at most one vertex in common.

(b) Any two edges $(u_1, u_2, ..., u_n)$ and $(v_1, v_2, ..., v_m)$ are equal if and only if (i) $m = n$ and (ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $1\leq i\leq n$.

For obvious reason, all vertices on an edge of a semigraph are considered to be adjacent to one another. Accordingly, the vertices are divided into four types namely end vertices, middle vertices, middle-end vertices and isolated vertices. Also the degree of a vertex in a semigraph has different forms in the context of semigraph depending upon its position on a particular edge or edges.

A semigraph *G* may be drawn as a figure in a plane using the set of points representing its vertices. An edge $E = (v_1, v_2, ..., v_n)$ is represented by a simple open Jordan curve (which may be drawn as a straight line as far as possible) whose end points are the end vertices of *E*. The middle vertex i.e., *m*-vertex of *E* which is not an *m*-vertex of another edge of *G* is denoted by a small circle placed on the curve in between its end vertices, in the order specified by *E*. The end vertex of an edge which is not an *m*-vertex of another edge is represented by a thick dot. If an *m*-vertex of an edge *E* is an end vertex of another edge E', we draw a short tangent to the circle at the end of the edge E' (for illustration, refer to Fig.1)·

The structure of an edge of a semigraph helps with emergence of new ideas like subedge and partial edge of edges. A **subedge** of an edge $E=(v_1,v_2,...,v_n)$ is a k-tuple $E'=(v_{i_1},v_{i_2},...,v_{i_k})$ where $1\le i_1<i_2<...<i_k\le n$ or $1\le i_k<i_{k-1}<...<i_1\le n$ and a **partial edge** of E is a (j - i + 1)-tuple $E(v_i,v_j)=(v_i,v_{i+1},...,v_j)$, where $1\le i<j\le n$ (or a (i - j + 1)-tuple $E(v_j,v_i)=(v_j,v_{j+1},...,v_i)$, where $1\le j<i\le n$). From this definition it is clear that every edge is a subedge (partial edge) of itself and a proper subedge is not an edge. For otherwise it would contradict the condition that two edges have at most one vertex in common.

There are four types of degree for a vertex in a semigraph as defined in [2] viz., degree, edge degree, adjacent degree and consecutive adjacent degree. Here, we give definition for adjacent degree only. An **adjacent degree** of a vertex v denoted by $\mathbf{deg_a}\boldsymbol{v}$ is the number of vertices adjacent to v.

Example 2.1 *A semigraph is displayed in Fig.1, where the vertices* v_1,v_3,v_7 *are end vertices,* v_4,v_5,v_6 *are the middle vertices and the vertices* v_2,v_8 *are middle-end vertices.* (v_3,v_5,v_6) *is a subedge of the edge* (v_3,v_4,v_5,v_6,v_7). *But* (v_4,v_5,v_6) *represents a partial edge of the edge* (v_3,v_4,v_5,v_6,v_7). *Also,* $\deg_a v_3=6$ *and* $\deg_a v_2=3$.

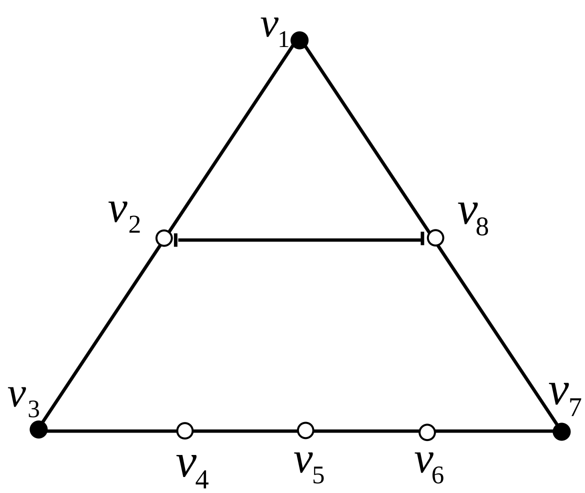

Fig.1

Similar to the concept of subgraph of a graph, we define subsemigraph of a semigraph depending on the concept of subedges of a semigraph. Thus, a semigraph $G'=(V',E')$ is a **subsemigraph** of a semigraph $G=(V,E)$ if $V'\subseteq V$ and the edges in G' are subedges of edges of G and it is called a **spanning subsemigraph** if, $V=V'$.

A spanning subsemigraph is called **saturated spanning subsemigraph** [3] of a semigraph if its edges are the same as the edges of the original semigraph. We illustrate this by an example given below.

Example 2.2 *Let G = (V, X) be a semigraph where* $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ *and* $X = \{(v_1, v_2, v_3), (v_1, v_4, v_5), (v_3, v_6), (v_5, v_6, v_7)\}$ *as shown in the Fig.2 given below.*

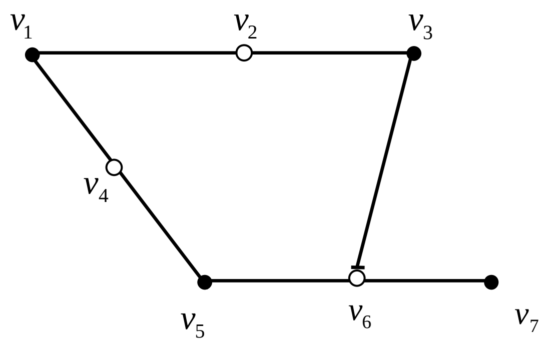

Fig.2

A saturated spanning subsemigraph of the above semigraph G is displayed below (Fig.3).

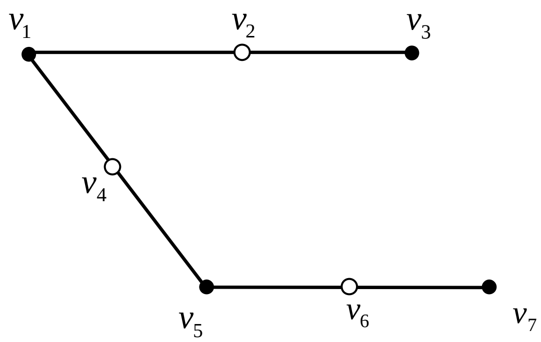

Fig.3

In [2], E. Sampathkumar defined adjacency graph associated with a semigraph. The **adjacency graph** G_a associated with a semigraph *G* = (*V*, *X*) has the same vertex set *V* of *G* with two vertices in G_a being adjacent if and only if they are adjacent in *G*.

Example 2.3 *The adjacency graph* G_a *associated with the semigraph given in Fig.2 is displayed below (Fig. 4).*

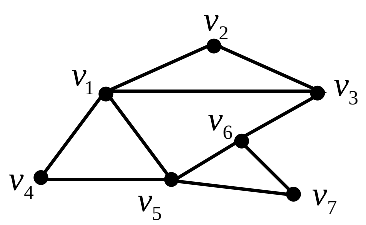

Fig. 4

The concepts of subedge and partial edge of a semigraph G motivate us for defining two different types of path. A path P is said to be an ***s*-path (strong path)** if any two consecutive vertices on it are also consecutive vertices of an edge of G otherwise, it is said to be a ***w*-path (weak path)**. Thus an *s*-path in a semigraph consists of edges and partial edges only. An *s*-path is an ***s*-cycle (strong cycle)** if its beginning and end vertices are same. A semigraph G is **connected** if there is a *w*-path or an *s*-path between any two vertices of G. However, we shall consider only *s*-paths here.

Example 2.4 *The Fig.5 displays a* $v_1 - v_9$ *s-path viz.,* $v_1 E_1 v_3 E_3 v_6 E_9 v_{10} E_7 v_9$ *of a semigraph, where* $E_1 = (v_1, v_2, v_3)$, $E_1 = (v_3, v_4, v_5, v_6)$, $E_9 = (v_6, v_{10})$ *and* $E_7 = (v_{10}, v_9)$. *This path can be written as* $v_1 v_2 v_3 v_4 v_5 v_6 v_{10} v_9$. *But* $v_1 v_3 v_4 v_6 v_{10} v_9$ *is a* $v_1 - v_9$ *w-path because it involves the subedges* (v_1, v_3) *and* (v_3, v_4, v_6).

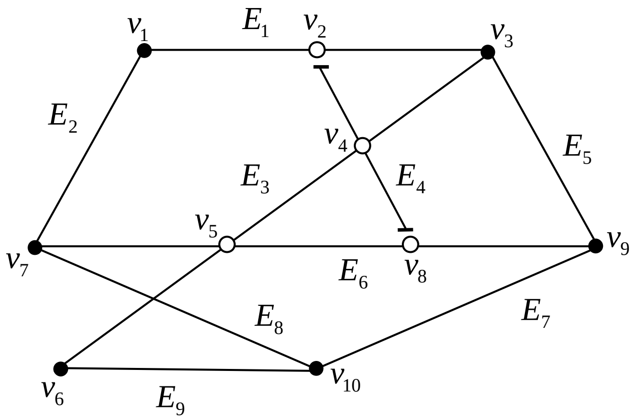

Fig.5

In [7], S.S. Kamath and R.S. Bhat defined adjacent neighbour set and consecutive neighbour set in a semigraph. For any vertex v in a semigraph $G = (V, X)$, the set $N_a(v) = \{x \in V \mid x$ is adjacent to $v\}$ is called an **adjacent neighbour set** of the vertex v and the set $N_{ca}(v) = \{x \in V \mid x$ is consecutive adjacent to $v\}$ is called a **consecutive adjacent neighbour set** of the vertex v. If $S \subseteq V$, then $N_a(S) = \cup_{v \in S} N_a(v)$. The set $N_a(S)$ is called the neighbourhood of the set S in G.

In a semigraph G, a vertex v and an edge E are said to **cover** each other if $v \in E$. A set S of vertices that covers all edges of a semigraph G is said to be a **vertex cover** for G. The **vertex covering number** $\alpha_0 = \alpha_0(G)$ is the minimum cardinality of a vertex cover for G. An edge cover for a semigraph G is a subset of $X=X(G)$ that covers all vertices of the semigraph $G = (V, X)$ and the minimum cardinality of such a subset is called the **edge covering number** of G. It is denoted by $\alpha_1(G)$.

A set S of vertices of a semigraph G is said to be **independent** if no edge is a subset of S. The maximum cardinality of such a set is the **vertex independence number** of G and it is denoted by $\beta_0 = \beta_0(G)$. Similarly, a set L of edges of a semigraph G is said to be **independent** if no two of the edges in L are adjacent. The **edge independence number** $\beta_1 = \beta_1(G)$ of G is the maximum cardinality of L.

The removal of a vertex v_i, $1 \le i \le n (n \ge 3)$ from an edge $E = (v_1, v_2, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n)$ of a semigraph *G* = (*V*, *X*) results in a subedge $E' = (v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n)$. If we remove a vertex *v* from *G* then we obtain a semigraph $G - v = (V', X')$ where $V' = V - \{v\}$ and the edge in X' are defined as follows:

If $E \in X$ and *E* does not contain *v* then $E \in X'$.
If $E \in X$ and *E* contains *v* then $E - v \in X'$ if and only if $|E| \ge 3$.

The removal of an edge *E* from *G* results in a semigraph $G' = (V', X')$ where $V = V'$ and $X' = X - E$.

A **cut vertex** in a semigraph *G* is the vertex whose removal increases the number of components of *G* and a **bridge** is an edge whose removal increases the number of components of *G*. If there is a bridge in a semigraph then, there exist two cut vertices incident with the bridge. A **non-separable** semigraph is the one which is connected, nontrivial and has no cut vertices.

Example 2.5 *In the semigraph shown in Fig. 6 below, the vertex v_1 is a cut vertex whereas v_2 is not a cut vertex. The edge $E_1 = (v_1, v_2, v_3)$ is a bridge but $E_2 = (v_2, v_4, v_5)$ is not a bridge.*

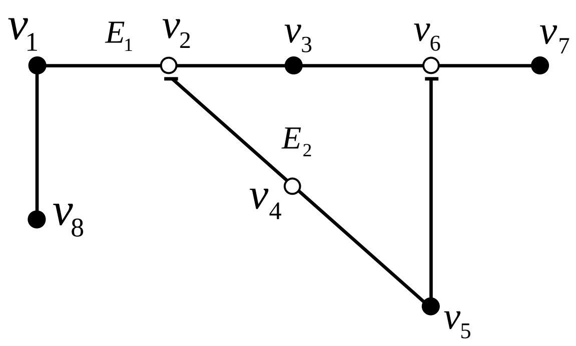

Fig. 6

Let *G* = (*V*, *X*) be a semigraph. A set $D \subseteq V$ is called **adjacent dominating set** (**ad-set**) if for every $v \in V - D$ there exist a $u \in D$ such that *u* is adjacent to *v* in *G*. The **adjacency domination number** is the minimum cardinality of an adjacent dominating set of *G*. It is denoted by $\gamma_a = \gamma_a(G)$. [7]

An **edge bipartite semigraph** is a semigraph which has no any odd *s*-cycles.

A **dendroid** is a connected semigraph without *s*-cycle. A dendroid is an edge bipartite semigraph and in fact, it is a generalization of a tree. The example of a dendroid is shown in the Fig. 7.

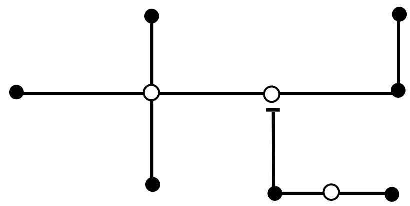

Fig. 7

Proposition 2.1 [2] *If T = (V, X) is a dendroid, then* $\alpha_0(T) = \beta_1(T)$.

For details on preliminaries about semigraphs we refer to [2].

3. The Matching in Semigraphs

In case of graphs, two edges are adjacent to each other if they have a common point. But in semigraphs there are different types of adjacency of edges, because the edges are n-tuple, which are defined as follows: Two edges in semigraph are said to be (i) *me*-adjacent if the common vertex is a middle vertex of one and an end vertex of the other, (ii) *mm*-adjacent if the common vertex is a middle vertex of both of the edges and (iii) *ee*-adjacent if the common vertex is an end vertex of both of the edges [2].

Two distinct edges E_1 and E_2 in a semigraph are said to be disjoint or adjacent according as $|E_1 \cap E_2| = 0$ or $|E_1 \cap E_2| = 1$.

Definition3.1 *A **matching** M in a semigraph G = (V, X) is the set of pair wise disjoint edges.*

Definition 3.2 *A vertex v of a semigraph G* = (*V*, *X*) *is said to be **saturated**, if there exists a matching M such that* $v \in E$ *for some* $E \in M$

Definition 3.3 *An edge E is said to be **saturated** by a matching M if* $E \in M$.

Definition 3.4 *A matching that saturates all the vertices is called a **perfect matching**.*

Clearly, a perfect matching and a 1_e-factor of a semigraph are one and the same thing [3].

Definition 3.5 *A matching M saturating the maximum number of edges of a semigraph G is called a **maximum matching**.*

A set consisting of $\beta_1(G)$ independent edges in *G* is a maximum matching of *G*.

We now illustrate some examples of matching and perfect matching.

Example 3.1 *There is a set of groups of workers* $\{W_1, W_2, ..., W_q\}$ *such that in each group there exist at least two workers and between any two groups there is at most one worker in common. Provided that each of these groups is formed according to the experience of its members the issue of determining the maximum number of groups to be involved in a certain number of works so that no worker is attached to more than two works at a time or the number of groups of workers to be involved in certain number of works so that all workers are involved in the groups with the obvious condition that no worker can do more than two works at a time reflects the example of a maximum matching or a perfect matching.*

Example 3.2 *There are p number of cities* (*or stations*) *and q number of passenger train routes* (*a route may be used to imply the passage of a train touching all the stations on its way or those stations on its way at which it has stoppages*) *through these cities such that between any two routes there exists at most one city in common and in any route there exist more than or equal to two cities. Such a network is clearly capable of describing various types of matchings of semigraphs.*

C. Berge [4] gives a characterization of maximum matching in graph and hypergraph. We give a similar result in semigraph. Before going to characterize the maximum matching we incorporate a definition followed by a theorem.

Definition 3.6 *A collection* $\{E_1, E_2, E_3, ..., E_n\}$ *of edges of a semigraph G = (V, X) is said to be a* ***chain*** *if (i)* $|E_i \cap E_{i+1}| = 1$ *and (ii)* $|E_i \cap E_j| = 0$, *for every* $i \neq j, j \neq i+1$, *i.e. no edges are adjacent except the consecutive ones.*

The length of a chain is odd (*respectively, even*) *if the number of edges involved in it is odd* (*respectively, even*).

Definition 3.7 *An* ***M-alternating chain*** *is a chain with respect to a matching M that alternates edges between those in M and those not in M. An M-alternating chain is* ***M-augmented*** *with respect to a matching M, if both the starting and ending edges are M-unsaturated edges.*

Clearly, the length of an M-augmenting chain relative to a matching M is always odd. It is always possible to obtain an s-path from a chain.

Proposition 3.1 *Let* $M_1 \Delta M_2 = (M_1 - M_2) \cup (M_2 - M_1)$ *be the symmetric difference of two matchings of semigraph G and H be the subsemigraph of G induced by* $M_1 \Delta M_2$. *Then the components of H either contain an s-cycle or an s-path with edges* (*or partial edges*) *alternately in* M_1 *and* M_2 *such that the starting and the ending vertices of this s-cycle or s-path are unsaturated in* M_1 *or* M_2.

Proof:

Let *v* be any vertex in *H*. Since M_1 and M_2 are matchings in *G*, there is at most one edge in M_1 or there is at most one edge in M_1 and at most one edge in M_2 incident with *v*. Hence the edge degree of *v* in *H* is either 1 or 2. So, every components of *H* is an *s*-

cycle or an *s*-path with edges (or partial edges) alternately in M_1 and M_2. Also the starting and ending vertices of this *s*-cycle or *s*-path are clearly unsaturated either in M_1 or M_2. ■

We now obtain a characterization of semigraph to have maximum matching.

Proposition 3.2 *A matching M in a semigraph G =* (*V*, *X*) *is a maximum matching if and only if G contains no M-augmenting chain.*

Proof:

Let *M* be a maximum matching in a semigraph *G* = (*V*, *X*) and let $\{E_1, E_2, E_3, ..., E_n\}$ be an *M*-augmenting chain in *G*. Then by definition, this chain is of odd length and the edges $E_1, E_3, ..., E_n$ are not in *M* whereas the edges $E_2, E_4, ..., E_{n-1}$ are in *M*. Also the edges $E_1, E_3, ..., E_n$ form a matching for *G*, whose length is |*M*|+1, |M| being the number of edges in M. This contradicts the maximum nature of *M*. Hence *G* has no *M*-augmenting chain.

Conversely suppose that *G* is without any *M*-augmenting chain. Let M' be another matching in *G* larger than *M*. Let $H = M \Delta M'$. Then, by proposition 3.1, we have an alternating chain which is an *s*-cycle or an *s*-path with more edges (or partial edges) in M' than edges in *M*. This chain can be an *s*-path only which starts and ends with edges in M' and so it is an *M*-augmenting chain in *G*. ■

We now derive a result relating a matching with a vertex cover of a semigraph.

Proposition 3.3 *Let M be a matching and C be a vertex cover of a semigraph G. Then* |*C*| ≥ |*M*|.

Proof:

The set C covers every edges of the semigraph G whereas the set M contains only the disjoint edges of G. Therefore, we have |*C*| ≥ |*M*|. ■

From the preceding result it follows immediately that $\alpha_0(G) \geq \beta_1(G)$, for any semigraph *G*. We now focus on a result for an edge bipartite semigraph *G*, analogue of which is the well known König's [6] theorem of graph theory.

Proposition 3.4 *Let G be any edge bipartite semigraph. Then the maximum size of a matching in G equals the minimum size of its vertex cover.*

Proof:

We prove the theorem by considering various possible cases of semigraphs as follows:

Case I: Suppose *G* is a connected semigraph without a cut vertex i.e. *G* is non-separable.

Every *s*-cycle in *G* must be even. We consider a longest *s*-cycle $C = E_1E_2E_3...E_{n-1}E_nE_{n+1}$ (*n* is an odd positive integer) in *G*. From this *s*-cycle, it is possible to find a longest *s*-path $P = E_1E_2E_3...E_{n-1}E_n$ in *G* of odd length from which we have a matching *M*. We construct the matching *M* by choosing the edges $E_1, E_3, E_5, ..., E_{n-2}, E_n$ from the *s*-path *P*. This *s*-path *P* is not *M*-augmented, as it starts and ends with the *M*-saturated edges (or partial edges). Therefore *M* is a maximum matching in *G* and $|M| = \frac{n+1}{2}$. Consequently, $\beta_1(G) = \frac{n+1}{2}$.

We now obtain a vertex covering set *S* from the *s*-cycle *C*, by choosing the common vertex from every pair (E_i, E_{i+1}) (i = 1,3,…,n) of the *s*-cycle. The cardinality of the set *S* is $\frac{n+1}{2}$ where *S* is the minimum vertex cover set for the edges of the *s*-cycle *C*. Therefore $|S| = \beta_1 = \frac{n+1}{2}$ and thus follows the proposition.

Case II: Suppose the semigraph *G* is disconnected so that *G* has at least two components. Applying Case I for each component of *G* we have the required result.

Case III: Suppose the semigraph *G* has a cut vertex *v*.

Let G_1 and G_2 be the two components of *G*, which are clearly edge bipartite semigraphs. Applying the Case I for each of them we obtain, $\beta_1(G_1) = \alpha_0(G_1)$ and $\beta_1(G_2) = \alpha_0(G_2)$. If *S* is a minimum vertex cover set of *G*,

$$\alpha_0(G) = |S| = \alpha_0(G_1) + \alpha_0(G_2) + |\{v\}|, \qquad (1)$$

where, *v* is the cut vertex of *G*.

If *M* is the maximum matching of *G*,

$$\beta_1(G) = |M| = \beta_1(G_1) + \beta_1(G_2) + 1 \qquad (2)$$

1 in (2) above corresponds to the edge considered for the matching of *G*.

Hence from (1) and (2), we have $\alpha_0(G) = \beta_1(G)$.

Case IV: Suppose the semigraph *G* has a bridge *E*.

Then we obtain at least one cut vertex in *G*. Applying the Case III we obtain the required result.

Case V: Suppose the semigraph *G* is a dendroid.

Then the result follows from the Proposition 2.1■

In graphs two maximum matchings have the same number of vertices. But the same is not true in case of semigraphs. However, in this connection, we have the following definition.

Definition 3.8 *A maximum matching M of a semigraph G is said to be* ***maximal vertex-saturated,*** *if it saturates maximum number of vertices of G among all the maximum matchings in G. We denote it by* M_{mvs} *and the number of vertices saturated by* M_{mvs} *is called the power of* M_{mvs}*, denoted by* $p(M_{mvs})$.

This is illustrated by the example given below (Fig.3). The edges of the semigraph shown in the Fig.8 are $E_1 = (v_1, v_2, v_3)$, $E_2 = (v_3, v_6, v_7)$, $E_3 = (v_2, v_4)$, $E_4 = (v_4, v_5)$ and $E_5 = (v_5, v_6)$. The matching $M_1 = \{E_1, E_5\}$ and $M_2 = \{E_3, E_5\}$ are both maximum matchings. But the number of vertices saturated by M_1 is more than those saturated by M_2.

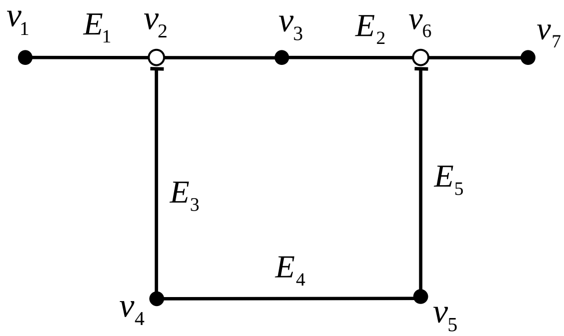

Fig.8

The three immediate results in this connection are produced below.

Proposition 3.5 *In a semigraph G = (V, X), let M be a maximal vertex-saturated matching. If* $p(M_{mvs}) = |V|$ *then the matching M is the perfect matching of G.*

Proof:

The proof is trivial since for a maximal vertex-saturated matching *M* with $p(M_{mvs}) = |V|$ the matching *M* saturates all the vertices of *G* and therefore it is clearly a perfect matching in *G*. ■

Proposition 3.6 *Let M be a maximum matching in a semigraph G. Then M is a maximal vertex-saturated matching if and only if*

$$\sum_{v \in M} |N_a(v)| > \sum_{v \in M'} |N_a(v)|$$

for any maximum matching M' *other than M in G.*

Proof:

Suppose *M* is a maximal vertex-saturated matching and M' is any other maximum matching of the semigraph *G*. Then there is at least one $M'-$unsaturated vertex in *G* which is saturated by *M*. So we have,

$$\sum_{v \in M} |N_a(v)| > \sum_{v \in M'} |N_a(v)|.$$

Conversely, suppose

$$\sum_{v \in M} |N_a(v)| > \sum_{v \in M'} |N_a(v)|$$

for any two maximum matching M and M' in G.

We are to show that M is M_{mvs}. Let us assume the contrary i.e., let M be not M_{mvs}. But the number of vertices saturated by M and M' is same. Thus, for subsemigraphs induced by the vertices which are saturated by M and M' respectively we shall have,

$$\sum_{v \in M} |N_a(v)| = \sum_{v \in M'} |N_a(v)|,$$

which contradicts our assumption. Hence follows the result. ■

Proposition 3.7 *Let M be a maximum matching in a semigraph G. Then M is maximal vertex-saturated matching if and only if*

$$\sum_{v \in M} |N_{ca}(v)| > \sum_{v \in M'} |N_{ca}(v)|$$

for any maximum matching M' other than M in G.

Proof:

Trivial.■

While studying the maximum matchings and maximal vertex saturated matchings for semigraphs we attempted to characterize them for (p, q) complete semigraphs, though without success. Particularly, it remains open to find the maximal vertex-saturated matching on a (p, q) complete semigraph G and its power $p(M_{mvs})$.

Similar results can be obtained in connection with the concept of perfect matching of semigraphs. We observe that, contrary to the cases of graphs, there are examples of perfect matchings (which are clearly 1_e-factors [3]) of semigraphs having distinct number of edges. To confirm our assertion, we require the following definition of minimal edge saturated matching for semigraphs.

Definition 3.9 *A perfect matching M of a semigraph G is said to be* ***minimal edge-saturated****, if it saturates the minimum number of edges of G among all perfect matchings of G and it is denoted by M_{mes}. The number of edges saturated by M_{mes} is called the power of M_{mes} which we denote by $p(M_{mes})$.*

We demonstrate the situation with the help of an example as shown in Fig. 9 below where, we have two perfect matchings $M=\{(v_1,v_2,v_3,v_4),(v_5,v_6,v_7,v_8)\}$ and $M'=\{(v_1,v_8),(v_2,v_7),(v_3,v_6),(v_4,v_5)\}$ out of which the matching M is minimal edge-saturated.

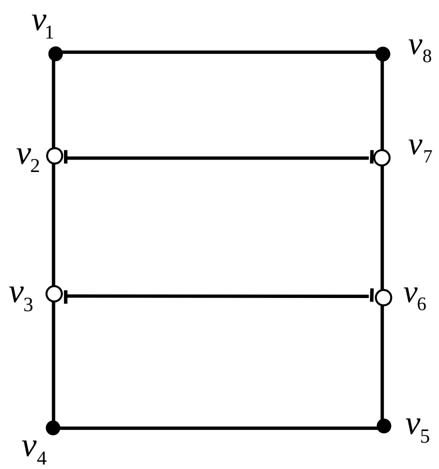

Fig. 9

Proposition 3.8 *In a semigraph G, a perfect matching M is minimal edge-saturated if and only if,* $\alpha_1(G)=|M|=P(M_{mes})$ *where* $\alpha_1(G)$ *denotes the edge covering number of G.*

Proof:

Suppose *M* is a minimal edge-saturated matching of a semigraph *G*. Then it is a perfect matching of G covering all of its vertices. Also it contains the least number of edges covering all vertices of *G*. Consequently, $\alpha_1(G)=|M|=P(M_{mes})$.

Conversely, suppose *M* is a perfect matching of *G* with $\alpha_1(G)=|M|=P(M_{mes})$. Thus, *M* is a perfect matching saturating minimum number of edges of *G*. Also, by definition of minimal edge-saturated matching, it is clear that *M* is minimal edge-saturated in *G*. ■

In case of an ordinary graph a maximum matching saturates largest number of its vertices. However, the same is not always true for semigraphs. In other words, a matching in a semigraph may saturate largest number of vertices though it may not be maximum one. Therefore, it is not out of context to formalize this situation in the form of a definition which may help characterization of distinguishing aspects of semigraphs. We like to name such a matching as an **optimum matching**.

Definition 3.10 *A matching M of a semigraph G is called an* ***optimum matching*** *if it has the smallest number of edges saturating the largest number of vertices of G.*

In the light of this definition, it follows that a minimal edge saturated matching of a semigraph is always an optimum matching.

The Fig.10 shows a semigraph in which the matching $M' = \{E_2, E_3, E_5\}$ is a maximum matching while the matching $M = \{E_1, E_5\}$ is not maximum. However, the number of vertices saturated by *M* is more than the number of vertices saturated by M'. Thus, the matching *M* is an optimum matching.

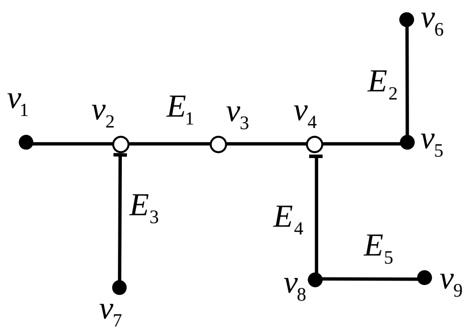

Fig.10

Application: The Example 3.2 mentioned above hints at scope for applications in networking problems particularly, in railway networks of a country. Perhaps, we may design a rail network to have maximum number of mutually disjoint routes in which we can provide trains to reach maximum number of cities (stations) running at the same time (corresponding to a maximal vertex saturated matching) or a rail network to have a minimum number of routes reaching maximum number of stations (corresponding to an optimum matching of a semigraph).

We establish the following property of an optimum matching in semigraphs.

Proposition 3.9 *Let* M_{op} *and M be a matching and maximum matching of a semigraph G respectively. Then* M_{op} *is an optimum matching of G if and only if*

$$|N_a(M_{op})| \geq |N_a(M)|.$$

Where $N_a(M_{op}) = \{\cup N_a(v) \mid v \in E, E \in M_{op}\}$ *and* $N_a(M) = \{\cup N_a(v) \mid v \in E, E \in M\}$

Proof:

Let M_{op} be an optimum matching of *G*. Then there is at least one vertex of *G* which is not saturated by *M*. So, we have,

$$|N_a(M_{op})| > |N_a(M)|.$$

If the maximum matching *M* is also a perfect matching of *G*, we have

$$|N_a(M_{op})| = |N_a(M)|.$$

Hence, $|N_a(M_{op})| \geq |N_a(M)|$.

Conversely, let M_{op} and M be a matching and maximum matching of a semigraph G respectively such that $|N_a(M_{op})| \geq |N_a(M)|$. Then from the definition of optimum matching it is clear that M_{op} is an optimum matching of G. ■

4. Relation between Domination and Matching in Semigraph

In this section, we introduce the concept of total adjacent domination and deduce some relations between the adjacent domination and the total adjacent domination in a particular type of matching in semigraphs.

The concept of total domination in graph was introduced by Cockayne et al. [10].

Definition 4.1 *A set of vertices in a semigraph G = (V, X) is said to be a* ***total adjacent dominating set (tad-set)*** *in G, if for every vertex of V is adjacent to a vertex in D. The* ***total adjacent domination number*** *of a semigraph is the minimum cardinality of a total adjacent dominating set in G. It is denoted by* $\gamma_{ta} = \gamma_{ta}(G)$.

In this case, we mention the following two results.

Theorem 4.1 [11] *For every graph G with no isolated vertex,* $\gamma(G) \leq \beta_1(G)$, *where* $\gamma(G)$ *and* $\beta_1(G)$ *are the domination number and edge independence number of G respectively.*

Theorem 4.2 [12] *For every k-regular graph G with* $k \geq 3$, $\gamma_t(G) \leq \beta_1(G)$, *where* $\gamma_t(G)$ *and* $\beta_1(G)$ *is the total domination number and edge independence number of G respectively.*

We now obtain the following results on total domination number of semigraphs.

Proposition 4.1 *For any semigraph* $G, \gamma_{ta}(G) = \gamma_t(G_a)$, *where* $\gamma_t(G_a)$ *is the total domination number of the adjacent graph* G_a *of G.*

Proof:

Trivial. ■

We now deduce the following proposition in connection with the set of maximum matching, adjacent domination number, total domination number and minimal edge-saturated matching.

Proposition 4.2 *Let the semigraph G =* (*V, X*) *without any isolated vertex and contain a minimal edge-saturated matching* M_{mes}. *Then* $\gamma_{ta}(G) \leq p(M_{mes})$, *where* $p(M_{mes})$ *denotes the power of minimal edge-saturated matching* M_{mes} *in G.*

Proof:

Let D be a minimal total adjacent dominating set of $G = (V, X)$. Since G is a semigraph having no isolated vertex, each vertex of G covers at least one edge of G and therefore for any $v_i \in D$ we have,

$$C_{v_i} = \{E_i \mid v_i \in E_i, E_i \in X\}.$$

We now construct a set M of edges of G by taking one edge E_i from each C_{v_i} such that the cardinality of E_i is maximum among all the edges in $C_{v_i}, v_i \in D$ and satisfying $|E_i \cap E_j| = 0$ for $i \neq j$. Then the set M is clearly a matching in G. We now consider the following cases.

Case 1: Let M is not a perfect matching of G. Since G contains a minimal edge-saturated matching, therefore it is possible to obtain a perfect matching from M by adding one or more edge to M. Thus we have $\gamma_{ta}(G) \leq p(M_{mes})$.
Case 2: Let M be a perfect matching such that M is a minimal edge-saturated matching. Then $\gamma_{ta}(G) \leq p(M_{mes})$.
Case 3: Let M be a perfect matching which is not a minimal edge-saturated matching. In that case, G contains more than one perfect matching. Then it is possible to find out a minimal edge-saturated matching. So, $\gamma_{ta}(G) \leq p(M_{mes})$. This completes the require result. ■

Proposition 4.3 *For any semigraph G without isolated vertex containing minimal edge-saturated matching* M_{mes}, $\gamma_{ta}(G) \leq \beta_1(G)$.

Proof:

Since G has a minimal edge-saturated matching M_{mes}, $M_{mes}(G) \leq \beta_1(G)$. Also from the above Proposition 4.2, we have $\gamma_{ta}(G) \leq p(M_{mes})$. Hence, $\gamma_{ta}(G) \leq \beta_1(G)$.■

In graph theory M. A. Henning et al. [12] successfully obtain the Theorem 4.2 which determines the relationship between total domination number and edge independence number (maximum matching number). We investigate the relationship between total adjacent domination number and edge independence number in semigraphs and obtain the following results.

Proposition 4.4 *Let G be a semigraph without isolated vertex having a minimal edge-saturated matching* M_{mes}. *Then,* $\gamma_a(G) \leq \beta_1(G)$ *and* $\gamma_a(G) \leq p(M_{mes})$.

Proof:

From the definition of adjacent domination and total adjacent domination in any semigraph G with on isolated vertex $\gamma_a(G) \leq \gamma_{ta}(G)$. Combining the result $\gamma_{ta}(G) \leq \beta_1(G)$ (Proposition 4.3) with the above inequality we have, $\gamma_a(G) \leq \beta_1(G)$.

Also, $\gamma_{ta}(G) \leq p(M_{mes})$ (Proposition 4.2) and $\gamma_a(G) \leq \gamma_{ta}(G)$ determines $\gamma_a(G) \leq p(M_{mes})$. Hence the proof is completed. ■

Proposition 4.5 [7] *For any semigraph G without isolated vertex containing p number of vertices,*

$$\frac{p}{\Delta_a + 1} \leq \gamma_a(G) \leq p - \beta_0(G).$$

Where Δ_a *denotes the maximum adjacent degree of G and* $\beta_0(G)$ *denotes the edge independence number.*

Corollary 4.1 *If any semigraph G without isolated vertex containing p number of vertices, contains a minimal edge saturated matching then we have*

$$\frac{p}{\Delta_a + 1} \leq p(M_{mes}) \leq p - \beta_0(G).$$

Proof:

Combining the Proposition 4.4 and Proposition 4.5 we obtain the require result. ■

Proposition 4.6 *For every graph G which contains a perfect matching,* $\gamma_t(G) \leq \beta_1(G)$. *Where* $\gamma_t(G)$ *and* $\beta_1(G)$ *are the total domination number and edge independence number of G.*

Proof:

Every graph being also a semigraph, the result follows immediately from Proposition 4.3. ■

5. Conclusion

The results and examples discussed in this paper clearly indicate wider scope of applicability of semigraphs in real situations in which the methods of ordinary graphs cannot be applied owing to its particular structure. Our future attempts will be to explore possibilities of some more such results.

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