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## $\alpha^{**}$ - CLOSED SETS IN TOPOLOGICAL SPACES

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**Abstract:** In this paper, we introduce a new class of sets namely,  $\alpha^{**}$  closed sets. Properties of this set are investigated and we introduce new spaces namely  $T_{\alpha^{**}}$  spaces,  $_{ag}T_{\alpha^{**}}$  spaces,  $_{gs}T_{\alpha^{**}}$  spaces,  $_{g^*}T_{\alpha^{**}}$  spaces and  $_{g^*}T_{\alpha^{**}}$  spaces.

**Key words:**  $\alpha^{**}$ -closed set;  $T_{\alpha^{**}}$  spaces,  $_{ag}T_{\alpha^{**}}$  spaces,  $_{gs}T_{\alpha^{**}}$  spaces,  $_{g^*}T_{\alpha^{**}}$  spaces and  $_{g^*}T_{\alpha^{**}}$  spaces.

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## 1. INTRODUCTION

Levine[7] introduced the class of g-closed set in 1970. Maki et al [9] defined  $\alpha$  g-closed sets in 1994. S.P. Arya and Tour [2] defined gs-closed sets in 1990. Dontchev [6] introduced gsp-closed sets. M.K.R.S Veerakumar [16] introduced and studied  $g^*$ -closed sets and  $g^*$ -continuity in topological spaces. The purpose of this paper is to introduce a new class of sets called  $\alpha^{**}$ -closed sets and  $T_{\alpha^{**}}$  spaces,  $_{ag}T_{\alpha^{**}}$  spaces,  $_{gs}T_{\alpha^{**}}$  spaces,  $_{g^*}T_{\alpha^{**}}$  spaces and  $_{g^*}T_{\alpha^{**}}$  spaces and investigate some of their properties.

## 1. PRELIMINARIES

Throughout this paper  $(X, \tau)$  represents a non-empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space  $(X, \tau)$ ,  $Cl(A)$ ,  $\alpha Cl(A)$  and  $int(A)$  denote the closure,  $\alpha$  closure and the interior of A respectively.

**Definition:** A subset A of a topology space  $(X, \tau)$  is said to be

1. semi-closed [8] if  $int(Cl(A)) \subseteq A$ .
2. g-closed [7] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and U is open in X.
3. gs-closed [2] if  $Scl(A) \subseteq U$  whenever  $A \subseteq U$  and U is open in X.

4. gp-closed[11] if  $Pcl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $X$ .
5.  $\alpha$ -closed[12] if  $Cl(int\ Cl(A)) \subseteq A$ .
6. semi-pre closed[1] if  $int(Cl(int(A))) \subseteq A$ .
7.  $g\alpha$ -closed[10] if  $\alpha Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\alpha$  open in  $X$ .
8.  $\alpha$  g-closed[9] if  $\alpha Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $X$ .
9. gsp-closed[6] if  $Spcl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $X$ .
10.  $g^*$ -closed [16] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $g$ -open in  $X$ .
11.  $(g\alpha)^*$ -closed[3] if  $\alpha Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $g\alpha$ -open in  $X$ .
12.  $\omega$ -closed[15] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is semi-open in  $X$ .
13.  $\omega$  g-closed[13] if  $cl(int\ A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $X$ .
14.  $\overline{\psi}$ -closed[18] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $sg$ -open in  $X$ .
15.  $\overline{\psi}^*$ -closed[18] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\overline{\psi}$ -open in  $X$ .
16.  $\alpha^*$ -closed[17] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\alpha$ -open in  $X$ .
17.  $g^{**}$ -closed[14] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $g^*$ -open in  $X$ .

**Definition:** A topological space  $(X, \tau)$  is said to be a

1.  $T_{1/2}$  space[7] if every  $g$ -closed set in it is closed.
2.  $T_b$  space[5] if every  $gs$ -closed set in it is closed.
3.  $T_d$  space[5] if every  $gs$ -closed set in it is  $g$ -closed.
4.  $T_{\alpha d}$  space[4] if every  $\alpha$   $g$ -closed set in it is  $g$ -closed.
5.  $T_{\alpha b}$  space[4] if every  $\alpha$   $g$ -closed set in it is closed.

### 3. Basic properties of $\alpha^{**}$ -closed sets

We introduce the following definition.

**Definition 3.1:** A subset  $A$  of a topology space  $(X, \tau)$  is said to be  $\alpha^{**}$ -closed if  $\text{Cl}(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\alpha^*$ -open.

**Proposition 3.2:** Every closed set is  $\alpha^{**}$ -closed.

Proof follows from the definition.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.3:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\emptyset, X, \{a\}\}$ . Then  $A = \{a, c\}$  is  $\alpha^{**}$ -closed but not closed in  $(X, \tau)$ .

**Proposition 3.4:** Every  $\alpha^{**}$ -closed set is a g-closed set.

**Proof:** Let  $A$  be  $\alpha^{**}$ -closed set. Let  $A \subseteq U$  where  $U$  is open. Then  $U$  is  $\alpha^*$ -open.

Therefore  $\text{Cl}(A) \subseteq U$ , since  $A$  is  $\alpha^{**}$ -closed. Hence  $A$  is a g-closed set.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.5:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ . Then  $A = \{a, b\}$  is g-closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Proposition 3.6:** Every  $\alpha^{**}$ -closed set is a gs-closed set.

**Proof:** Let  $A$  be a  $\alpha^{**}$ -closed set. Let  $A \subseteq U$  where  $U$  is open. Then  $U$  is  $\alpha^*$ -open and  $\text{Cl}(A) \subseteq U$ , since  $A$  is  $\alpha^{**}$ -closed.  $\text{Scl}(A) \subseteq \text{Cl}(A)$ . Therefore  $\text{Scl}(A) \subseteq U$ , whenever  $A \subseteq U$  &  $U$  is open. Therefore  $A$  is a gs-closed set

The following example supports that a gs-closed set need not be  $\alpha^{**}$ -closed in general.

**Example 3.7:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ . Then  $A = \{c\}$  is a gs-closed set but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Proposition 3.8:** Every  $\alpha^{**}$ -closed set is a gp-closed set.

**Proof:** Let  $A$  be a  $\alpha^{**}$ -closed set. Let  $A \subseteq U$  where  $U$  is open. since  $U$  is open,  $U$  is  $\alpha^*$ -open. Hence  $Cl(A) \subseteq U$ , since  $A$  is  $\alpha^{**}$ -closed set.  $Pcl(A) \subseteq Cl(A)$ . Therefore  $Pcl(A) \subseteq U$ , whenever  $A \subseteq U$  &  $U$  is open and hence  $A$  is a gp-closed set.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.9:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{c\}$  is gp-closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Proposition 3.10:** Every  $\alpha^{**}$ -closed set is  $\alpha$ g-closed.

**Proof:** Let  $A$  be a  $\alpha^{**}$ -closed set. Let  $A \subseteq U$ , where  $U$  is open. since  $U$  is open,  $U$  is  $\alpha^*$ -open. Hence  $Cl(A) \subseteq U$ , since  $A$  is  $\alpha^{**}$ -closed set.  $\alpha Cl(A) \subseteq Cl(A)$  and hence  $A$  is a  $\alpha$ g-closed set.

The following example support that a  $\alpha$ g-closed set need not be  $\alpha^{**}$ -closed set in general.

**Example 3.11:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{c\}$  is a  $\alpha$ g-closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Proposition 3.12:** Every  $\alpha^{**}$ -closed set is a gsp-closed set.

Proof follows from the definitions.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.13:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{a, b\}$  is a gsp-closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Proposition 3.14:** Every  $\alpha^{**}$ -closed set is a  $\omega$ g-closed set.

**Proof:** Let  $A$  be a  $\alpha^{**}$ -closed set. Let  $A \subseteq U$  and  $U$  be open. Since  $U$  is open,  $U$  is  $\alpha^*$ -open. Hence  $Cl(A) \subseteq U$ , since  $A$  is  $\alpha^{**}$ -closed set  $Cl(int(A)) \subseteq Cl(A) \subseteq U$  & hence  $cl(int(A)) \subseteq U$ , whenever  $A \subseteq U$  &  $U$  is open.  $\therefore A$  is a  $\omega$ g-closed set.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.15:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{c\}$  is a  $\omega$ g-closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Proposition 3.16:** Every  $\alpha^*$ -closed set is a  $\alpha^{**}$ -closed set.

proof follows from the definitions.

The converse of the above need not be true.

**Example 3.17:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $A = \{b\}$  is a  $\alpha^{**}$ -closed but not  $g^*$ -closed in  $(X, \tau)$ .

**Remark 3.18:**  $\alpha^{**}$ -closedness is independent of  $g^*$ -closedness.

**Example 3.19:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{c\}$  is  $g^*$ -closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Example 3.20:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{b\}, \{b, c\}\}$ . Then  $A = \{a, b\}$  is  $\alpha^{**}$ -closed set but not  $g^*$ -closed in  $(X, \tau)$ .

Hence  $\alpha^{**}$ -closedness is independent of  $g^*$ -closedness.

**Remark 3.21:**  $\omega$ -closedness is independent of  $\alpha^{**}$ -closedness.

**Example 3.22:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{b\}$  is a  $\omega$ -closed set but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Example 3.23:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $B = \{b\}$  is  $\alpha^{**}$ -closed but not  $\omega$ -closed in  $(X, \tau)$ .

$\therefore \omega$ -closedness is independent of  $\alpha^{**}$ -closedness.

**Remark 3.24:**  $\alpha$ -closedness is independent of  $\alpha^{**}$ -closedness.

**Example 3.25:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $A = \{a, b\}$  is  $\alpha^{**}$ -closed but not  $\alpha$ -closed in  $(X, \tau)$ .

**Example 3.26:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{a, c\}\}$ . Then  $B = \{c\}$  is  $\alpha$ -closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

$\therefore \alpha$ -closedness is independent of  $\alpha^{**}$ -closedness.

**Remark 3.27:**  $\alpha^*$ -closedness is independent of  $\alpha^{**}$ -closedness.

**Example 3.28:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $A = \{a, b\}$  is  $\alpha^{**}$ -closed but not  $\alpha^*$ -closed in  $(X, \tau)$ .

**Example 3.29:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $B = \{b\}$  is  $\alpha^*$ -closed set but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

Hence  $\alpha^*$  closedness is independent of  $\alpha^{**}$  closedness.

**Proposition 3.30:** Every  $\overline{\psi}$ -closed set is  $\alpha^*$  closed.

Proof follows from the definitions.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.31:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{b\}$  is  $\alpha^*$ -closed but not  $\overline{\psi}$ -closed in  $(X, \tau)$ . **Proposition 3.32:** Every  $\alpha^{**}$ -closed set is  $\overline{\psi}^*$ -closed.

Proof follows from the definitions.

The converse of the above proposition need not true in general as seen in the following example.

**Example 3.33:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $A = \{b\}$  is a  $\overline{\psi}^*$ -closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

**Remark 3.34:**  $(g\alpha)^*$  closedness is independent of  $\alpha^{**}$  closedness.

**Example 3.35:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $A = \{a, b\}$  is  $\alpha^{**}$ -closed but not  $(g\alpha)^*$ -closed in  $(X, \tau)$ .

**Example 3.36:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{c\}, \{a, c\}\}$ . Then  $B = \{a\}$  is  $(g\alpha)^*$ -closed but not  $\alpha^{**}$ -closed in  $(X, \tau)$ .

Hence  $(g\alpha)^*$  closedness is independent of  $\alpha^{**}$  closedness.

**Proposition 3.37:** If  $A$  is both  $\alpha^*$ -open and  $\alpha^{**}$ -closed set of  $(X, \tau)$  then  $A$  is a closed set.

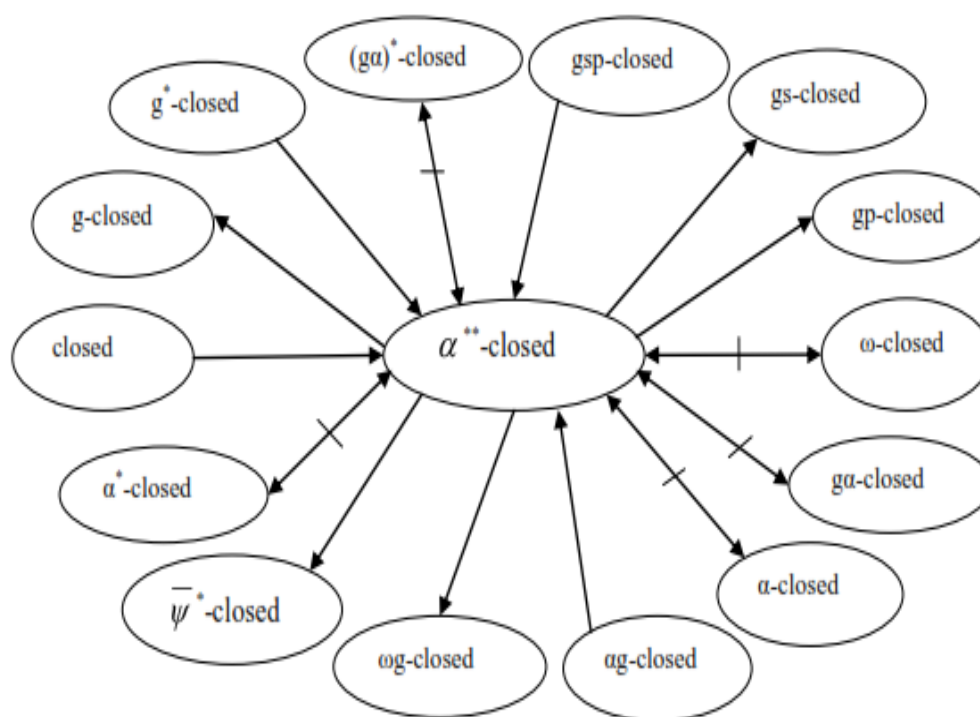
**Proposition 3.38:**  $A$  is  $\alpha^{**}$ -closed set of  $(X, \tau)$  if and only if  $\text{Cl}(A) \setminus A$  does not contain any non-empty  $\alpha^*$ -closed set.

**Proof: Necessity:** Let  $F$  be a  $\alpha^*$ -closed set of  $(X, \tau)$  such that  $F \subseteq \text{Cl}(A) \setminus A$ . Then  $A \subseteq X \setminus F$ .  $A$  is an  $\alpha^{**}$ -closed set and  $X \setminus F$  is  $\alpha^*$ -open, such that  $A \subseteq X \setminus F$ . Then  $\text{Cl}(A) \subseteq X \setminus F$ , since  $A$  is  $\alpha^{**}$ -closed.  $F \subseteq X \setminus \text{Cl}(A)$ . Hence  $F \subseteq ((X \setminus \text{Cl}(A)) \cap ((\text{Cl}(A) \setminus A)) = \Phi$ . Therefore  $F = \Phi$ .

**Sufficiency:** Let  $A$  be a subset of  $(X, \tau)$  such that  $\text{Cl}(A) \setminus A$  does not contain any non-empty  $\alpha^*$ -closed set. Let  $U$  be a  $\alpha^*$ -open set of  $(X, \tau)$  such that  $A \subseteq U$ . If  $\text{Cl}(A) \not\subseteq U$ , then  $\text{Cl}(A) \cap U^c \neq \Phi$  and  $\text{Cl}(A) \cap U^c$  is  $\alpha^*$ -closed set. Therefore  $\Phi \neq \text{Cl}(A) \cap U^c \subseteq \text{Cl}(A) \setminus A$ . Therefore  $\text{Cl}(A) \setminus A$  contains a non-empty  $\alpha^*$ -closed set which is a contradiction. Therefore  $\text{Cl}(A) \subseteq U$  and hence  $A$  is  $\alpha^{**}$ -closed.

**Proposition 3.39:** If  $A$  is an  $\alpha^{**}$ -closed set  $(X, \tau)$  such that  $A \subseteq B \subseteq \text{Cl}(A)$ , then  $B$  is an  $\alpha^{**}$ -closed set of  $(X, \tau)$ .

**Remark 3.40:** Thus we have the following diagram.



Where  $A \longrightarrow B$  (res.  $A \longleftrightarrow B$ ) represents  $A$  implies  $B$  but not conversely (resp.  $A$  and  $B$  are independent).

#### 4. Applications of $\alpha^{**}$ -closed sets

In this section we introduce five new spaces namely  $T_{\alpha^{**}, ag} T_{\alpha^{**}, gs} T_{\alpha^{**}, g} T_{\alpha^{**}}$  and  $g^* T_{\alpha^{**}}$ .

**Definition 4.1:** A space  $(X, \tau)$  is called a  $T_{\alpha^{**}}$  space if every  $\alpha^{**}$ -closed set is closed.

**Theorem 4.2:** Every  $T_{1/2}$  space is a  $T_{\alpha}^{**}$  space.

Proof follows from the definitions.

The converse of the above theorem need not true in general as seen in the following example.

**Example 4.3:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $(X, \tau)$  is a  $T_{\alpha}^{**}$  space.  $A = \{a, b\}$  is  $g$ -closed but not a closed set. Therefore  $(X, \tau)$  is not a  $T_{1/2}$  space.

**Theorem 4.4:** Every  $T_b$  space is a  $T_{\alpha}^{**}$  space.

**Proof:** Let  $(X, \tau)$  be a  $T_b$  space. Let  $A$  be a  $\alpha^{**}$ -closed set. Then  $A$  is  $g$ s-closed. Since  $(X, \tau)$  is a  $T_b$  space,  $A$  is closed. Hence the space  $(X, \tau)$  is a  $T_{\alpha}^{**}$  space.

The converse of the above theorem need not true in general as seen in the following example.

**Example 4.5:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $(X, \tau)$  is a  $T_{\alpha}^{**}$  space.  $A = \{c\}$  is  $g$ s-closed but not a closed set. Therefore  $(X, \tau)$  is not a  $T_b$  space.

**Theorem 4.6:** A space which is both  $T_{1/2}$  and  $T_d$  is  $T_{\alpha}^{**}$  space.

**Theorem 4.7:** A space which is both  $T_d$  and  $T_{1/2}$  is a  $T_{\alpha}^{**}$  space.

**Proof:** Let  $A$  be a  $\alpha^{**}$ -closed set. Then  $A$  is a  $\alpha$   $g$ -closed set. Since  $X$  is a  $T_d$  space,  $A$  is  $g$ -closed. Since the space is a  $T_{1/2}$  space,  $A$  is closed. Hence  $(X, \tau)$  is a  $T_{\alpha}^{**}$  space.

**Theorem 4.8:** Every  $T_b$  space is a  $T_{\alpha}^{**}$  space but not conversely.

**Example 4.9:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $(X, \tau)$  is a  $T_{\alpha}^{**}$  space.  $A = \{c\}$  is  $\alpha$   $g$ -closed set but not a closed set. Therefore  $(X, \tau)$  is not a  $T_b$  space.

We introduce the following definition.

**Definition 4.10:** A space  $(X, \tau)$  is called a  $g_s T_{\alpha}^{**}$  space if every  $g$ s-closed set is a  $\alpha^{**}$ -closed set.

**Theorem 4.11:** Every  $T_b$  space is a  $g_s T_{\alpha}^{**}$  space but not conversely.

**Example 4.12:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $(X, \tau)$  is a  $g_s T_{\alpha}^{**}$  space.  $A = \{b\}$  is a  $g$ s-closed set but not a closed set. Therefore  $(X, \tau)$  is not a  $T_b$ -space.



**Theorem 4.13:** A space which is both  $T_{\alpha^{**}}$  and  $_{gs}T_{\alpha^{**}}$  is a  $T_b$ -space.

**Proof:** Let  $A$  be a  $_{gs}$ -closed set of  $(X, \tau)$ . Then, since  $(X, \tau)$  is a  $_{gs}T_{\alpha^{**}}$ -space,  $A$  is  $\alpha^{**}$ -closed. & Since  $(X, \tau)$  is a  $T_{\alpha^{**}}$ -space,  $A$  is closed. Therefore  $(X, \tau)$  is a  $T_b$ -space.

**Theorem 4.14:** Every  $_{gs}T_{\alpha^{**}}$ -space is a  $T_d$ -space.

The converse of the above need not be true.

**Example 4.15:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $(X, \tau)$  is a  $T_d$ -space.  $A = \{b\}$  is  $_{gs}$ -closed set but not a  $\alpha^{**}$ -closed set. Therefore  $(X, \tau)$  is not a  $_{gs}T_{\alpha^{**}}$ -space.

**Remark 4.16:**  $T_{\alpha^{**}}$ -ness is independent from  $_{gs}T_{\alpha^{**}}$ -ness.

**Example 4.17:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $(X, \tau)$  is a  $_{gs}T_{\alpha^{**}}$ -space.  $A = \{b\}$  is  $\alpha^{**}$ -closed but not closed. Hence  $(X, \tau)$  is not a  $T_{\alpha^{**}}$ -space.

**Example 4.18:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $(X, \tau)$  is a  $T_{\alpha^{**}}$ -space.  $B = \{b\}$  is  $_{gs}$ -closed but not  $\alpha^{**}$ -closed. Hence  $(X, \tau)$  is not a  $_{gs}T_{\alpha^{**}}$ -space.

$\therefore T_{\alpha^{**}}$ -ness is independent from  $_{gs}T_{\alpha^{**}}$ -ness.

We introduce the following definition.

**Definition 4.19:** A space  $(X, \tau)$  is called a  $_{ag}T_{\alpha^{**}}$ -space if every  $\alpha$ -g-closed set is a  $\alpha^{**}$ -closed set.

**Theorem 4.20:** Every  $_aT_b$ -space is a  $_{ag}T_{\alpha^{**}}$ -space but not conversely.

**Example 4.21:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}\}$ . Then  $(X, \tau)$  is a  $_{ag}T_{\alpha^{**}}$ -space.  $A = \{b\}$  is  $\alpha$ -g-closed but not a closed. Therefore  $(X, \tau)$  is not a  $_aT_b$ -space.

**Theorem 4.22:** A space which is both  $T_{\alpha^{**}}$  and  $_{ag}T_{\alpha^{**}}$  is a  $_aT_b$ -space.

**Theorem 4.23:** Every  $_{ag}T_{\alpha^{**}}$  is a  $_aT_d$ -space but not conversely.

**Example 4.24:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$ . Then  $(X, \tau)$  is a  $_aT_d$ -space.  $A = \{b\}$  is  $\alpha$ -g-closed but not  $\alpha^{**}$ -closed. Hence  $(X, \tau)$  is not a  $_{ag}T_{\alpha^{**}}$ -space.

**Remark 4.25:**  $T_{\alpha^{**}}$ ness is independent from  $_{ag}T_{\alpha^{**}}$ ness.

**Example 4.26:** Let  $X=\{a,b,c\}$ ,  $\tau=\{\Phi, X, \{a\}, \{b,c\}\}$ . Then  $(X, \tau)$  is a  $T_{\alpha^{**}}$ space.  $A=\{c\}$  is  $\alpha$  g closed but not  $\alpha^{**}$ -closed. Therefore  $(X, \tau)$  is not a  $_{ag}T_{\alpha^{**}}$ space.

**Example 4.26:** Let  $X=\{a,b,c\}$ ,  $\tau=\{\Phi, X, \{a\}\}$ . Then  $(X, \tau)$  is a  $_{ag}T_{\alpha^{**}}$ space.  $A=\{b\}$  is  $\alpha^{**}$ -closed but not closed. Hence  $(X, \tau)$  is not a  $T_{\alpha^{**}}$ space.

$\therefore T_{\alpha^{**}}$ ness is independent from  $_{ag}T_{\alpha^{**}}$ ness.

We now introduce the following definition.

**Definition 4.28:** A space  $(X, \tau)$  is called a  $_{g}T_{\alpha^{**}}$ space if every g-closed set is a  $\alpha^{**}$ -closed set.

**Theorem 4.29:** Every  $T_{1/2}$  space is a  $_{g}T_{\alpha^{**}}$ space but not conversely.

**Example 4.30:** Let  $X=\{a,b,c\}$ ,  $\tau=\{\Phi, X, \{a\}\}$ . Then  $(X, \tau)$  is  $_{g}T_{\alpha^{**}}$ space.  $A=\{b\}$  is g-closed but not a closed set. Therefore  $(X, \tau)$  is not a  $T_{1/2}$  space.

**Theorem 4.31:** A space which is both  $T_{\alpha^{**}}$  and  $_{g}T_{\alpha^{**}}$  is a  $T_{1/2}$  space.

**Theorem 4.32:** Every  $_{ag}T_{\alpha^{**}}$ space is a  $_{g}T_{\alpha^{**}}$ space but not conversely.

**Example 4.33:** Let  $X=\{a,b,c\}$ ,  $\tau=\{\Phi, X, \{b\}, \{b,c\}\}$ . Then  $(X, \tau)$  is a  $_{g}T_{\alpha^{**}}$ space.  $A=\{c\}$  is  $\alpha$  g closed but not  $\alpha^{**}$ -closed. Therefore  $(X, \tau)$  is not a  $_{ag}T_{\alpha^{**}}$ space.

**Remark 4.34:**  $T_{\alpha^{**}}$ ness is independent from  $_{g}T_{\alpha^{**}}$ ness.

**Example 4.35:** Let  $X=\{a,b,c\}$ ,  $\tau=\{\Phi, X, \{a\}, \{b,c\}\}$ . Then  $(X, \tau)$  is a  $T_{\alpha^{**}}$  space.  $A=\{c\}$  is g-closed but not  $\alpha^{**}$ -closed. Therefore  $(X, \tau)$  is not a  $_{g}T_{\alpha^{**}}$ space.

**Example 4.36:** Let  $X=\{a,b,c\}$ ,  $\tau=\{\Phi, X, \{a\}\}$ . Then  $(X, \tau)$  is  $_{g}T_{\alpha^{**}}$ space.  $A=\{b\}$  is  $\alpha^{**}$ -closed but not closed. Therefore  $(X, \tau)$  is not a  $T_{\alpha^{**}}$ space.

**Theorem 4.37:** Every  $_{gs}T_{\alpha^{**}}$ space is  $_{g}T_{\alpha^{**}}$ space.

The converse of the above theorem need not true in general as seen in the following example.

**Example 4.38:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{b\}, \{a, b\}\}$ . Then  $(X, \tau)$  is a  ${}_g T_{\alpha}^{**}$  space.  $A = \{a\}$  is a  $g_s$ -closed set but not a  $\alpha^{**}$  closed set. Therefore  $(X, \tau)$  is not a  ${}_g T_{\alpha}^{**}$  space.

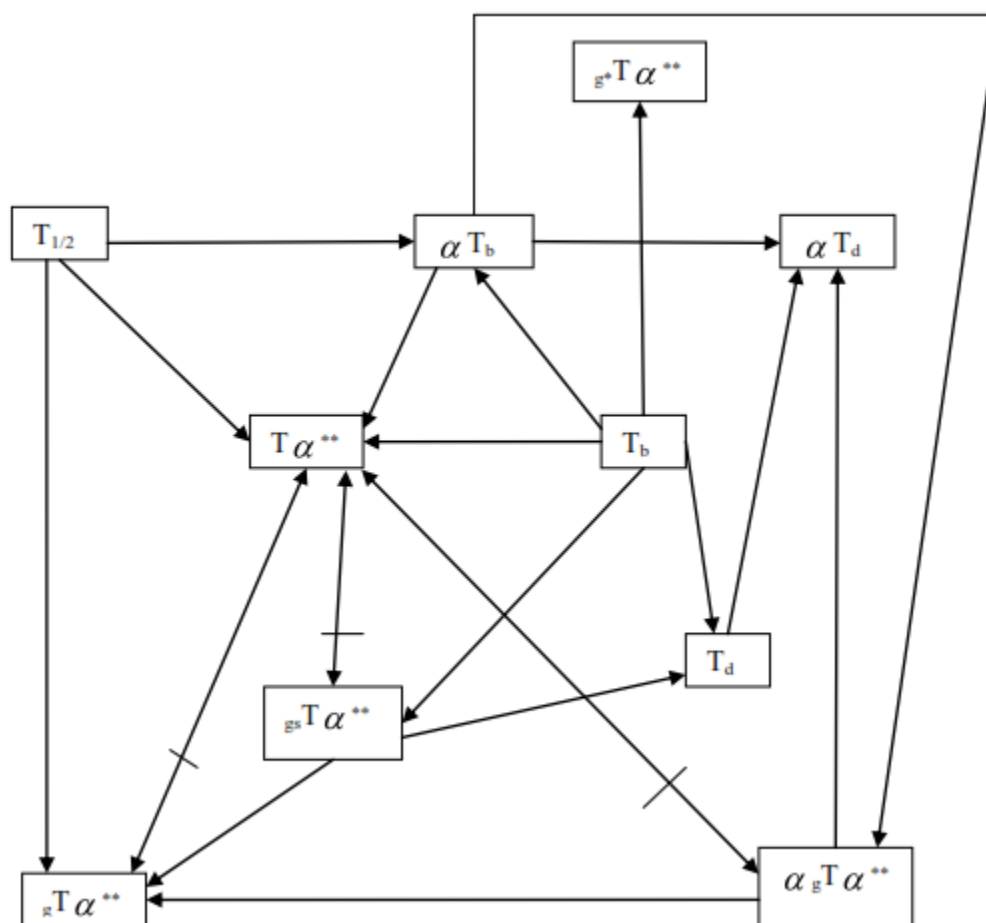
We now introduce the following definition.

**Definition 4.39:** A space  $(X, \tau)$  is called a  ${}_g T_{\alpha}^{**}$  space if every  $\alpha^{**}$ -closed set is  $g^*$ -closed.

**Theorem 4.40:** Every  $T_b$  space is a  ${}_g T_{\alpha}^{**}$  space but not conversely.

**Example 4.41:** Let  $X = \{a, b, c\}$ ,  $\tau = \{\Phi, X, \{a\}, \{a, c\}\}$ . Then  $(X, \tau)$  is a  ${}_g T_{\alpha}^{**}$  space.  $A = \{c\}$  is a  $g_s$ -closed set but not a closed set. Therefore  $(X, \tau)$  is not a  $T_b$ -space.

**Remark 4.42:** Thus we have the following diagram.



where  $A \longrightarrow B$  (resp.  $A \longleftrightarrow B$ ) represents  $A$  implies  $B$  but not conversely (resp.  $A$  and  $B$  are independent)

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