Separation axioms Via $g^{**}-open$ sets in topological spaces and ideal topological spaces

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Abstract: In this paper, we introduce g**-neighbourhood, g**-limit point, $g^{**}cl(A)$, $g^{**}-multiplicative$, $g^{**}-$ interior point, $g^{**}\text{int}(A)$, g**-resolute, g**-additive, g**-discrete, g**convergence. The separation axioms via g**-open sets are discussed in topological spaces and ideal topological spaces.

Key words: g**neighbourhood, g**limit point, $g^{**}cl(A)$, $g^{**}-multiplicative$, g^{**} interior point, $g^{**}\text{int}(A)$, g**-resolute, g**-additive, g**-discrete, g**-convergence, g**-T_0 space, g**- T_0 modulo *I* space, space g**- T_1 space g**- T_1 modulo *I* space, g**- T_2 space and g**- T_2 modulo *I* space,.

1. Introduction

Levine **[1]** introduced the class of g-closed sets in 1970 and M.K.R.S. Veerakumar**[5]** introduced g*-closed sets in 1991. Ideal topological spaces have been first introduced by K. Kuratowski **[2]** in 1930. In this paper we generalize the traditional separation axioms via g**-open sets.

2. Preliminaries

Definition 2.1: A subset A of a topological space(X, τ) is called

1) *generalized closed* (briefly *g-closed*)**[1]** if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
2) *generalized star closed* (briefly *g*-closed*)**[5]** if cl(A) ⊆ U whenever A ⊆ U and U is g- open in (X, τ).

3) *generalized star star closed* (briefly *g**-closed*)**[4]** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g*- open in (X, τ).

Definition 2.2: A function f **:** (X, τ) → (Y, σ) is called

1) g**-irresolute **[4]** if $f^{-1}(V)$ is a g**-closed set of (X, τ) for every g**-closed set V of (Y, σ).

2) g**-continuous **[4]** if $f^{-1}(V)$ is a g**-closed set of (X,τ) for every closed set V of (Y,σ).

Definition 2.3: An ideal**[2]** I on a non empty set X is a collection of subsets of X which satisfies the following properties.(i) $A\in I$, $B\in I \Rightarrow A\cup B\in I$ (ii) $A\in I$, $B\subset A \Rightarrow B\in I$ .A topological space (X,τ) with an ideal I on X is called an ideal topological space and is denoted by (X,τ,I).

3.gcl(A) and g**int(A)**

Definition 3.1: Let (X,τ) be a topological space and $x\in X$. Every $g^{**}-open$ set containing x is said to be a $g^{**}-neighbourhood$ of x.

Definition 3.2: Let A be a subset of *x*. A point $x\in X$ is said to be a $g^{**}-$ limit point of A if every $g^{**}-neighbourhood$ of x contains a point of A other than x.

The set of all $g^{**}-$ limit points of A is denoted by the symbol A'.

Definition 3.3: Let A be a subset of a topological space (X,τ). $g^{**}cl(A)$ is defined to be the intersection of all $g^{**}-closed$ sets containing A.

Note: **[3]** $g^{**}cl(A)$ need not be $g^{**}-closed$, since intersection of $g^{**}-closed$ sets need not be $g^{**}-closed$. But if A is $g^{**}-closed$ then $g^{**}cl(A)=A$.

Definition 3.4: A topological space (X,τ) is said to be $g^{**}-multiplicative$ if arbitrary intersection of $g^{**}-closed$ sets is $g^{**}-closed$. Equivalently arbitrary union of $g^{**}-open$ sets is $g^{**}-open$.

Note: If (X,τ) is $g^{**}-multiplicative$ then $A = g^{**}cl(A)$ if and only if A is $g^{**}-closed$.

Theorem 3.5: Let A be a subset of a topological space (X,τ). Then $g^{**}cl(A) = A \cup A'$.

Proof: $g^{**}cl(A) =$ intersection of all $g^{**}-closed$ sets containing A. Therefore $A \subseteq g^{**}cl(A)$. Let $x \in A'$ and suppose $x \notin g^{**}cl(A)$, then there exists $g^{**}-closed$ set F containing A such that $x \notin F$. Then $X - F$ is a $g^{**}-open$ set and $x \in X - F$. Therefore $(X - F) \cap (A - \{x\}) \neq \phi$ which is not true. Therefore $x \in g^{**}cl(A)$. Therefore $A \cup A' \subseteq g^{**}cl(A)$ .Let $x \in g^{**}cl(A)$ and $x \notin A$. Suppose $x \notin A'$ then there exists a $g^{**}-neighbourhood$ U of x such that $U \cap A = \phi$. Therefore $A \subseteq X - U$ which is $g^{**}-closed$ containing x and $x \notin X - U$, which is a contradiction. Therefore $g^{**}cl(A) \subseteq A \cup A'$. Hence $g^{**}cl(A) = A \cup A'$.

Theorem 3.6: Let (X,τ) be a $g^{**}-multiplicative$ space then a subset A of X is $g^{**}-closed$ if and only if $A \supseteq A'$.

Proof: By theorem (3.5), A is g**-closed if and only if $A = A \cup A' \Leftrightarrow A' \subseteq A$.

Definition 3.7: Let (X,τ) be a topological space and A be a subset of X. A point $x \in A$ is said to be g^{**} interior point of A if there exists $g^{**}-open$ set U such that $x \in U \subseteq A$.

Definition 3.8: Let A be a subset of a topological space (X,τ). $g^{**}\text{int}(A)$ is defined to be the union of all $g^{**}-open$ sets contained in A.

Note: 1. Obviously $g^{**}\text{int}(A)$ is the set of all g^{**} interior point of A.

2. $g^{**}\text{int}(A)$ need not be $g^{**}-open$ but if A is $g^{**}-open$ then $g^{**}\text{int}(A) = A$.

3. If (X,τ) is a $g^{**}-multiplicative$ space then $A = g^{**}\text{int}(A)$ if and only if A is $g^{**}-open$.

Theorem 3.9: A subset of a topological space (X,τ) is $g^{**}-open$ if and only if every point $x \in A$ is a g^{**} interior point of A .That is $A \subseteq g^{**}\text{int}(A)$

Proof: Necessity: Let A be $g^{**}-open$ and $x \in A$. Then X – A is $g^{**}-closed$ *and* $x \notin X - A$ Then $x \notin (X - A)'$ and hence there exists a $g^{**}-open$ set U such that $x \in U$ and $U \cap (X - A) = \varphi. \therefore U \subseteq A$ which proves that $x \in g^{**}\text{int}(A)$

Sufficiency: Let $A \subseteq g^{**}\text{int}(A)$. Let us prove that X – A is g**-closed. Let $x \in (x - A)'$. Suppose $x \notin X - A$ then $x \in A$. Therefore there exists g**-open set U such that $x \in U \subseteq A$ which is a contradiction to the fact that $x \in (x - A)'$. Therefore X –A is g**-closed and hence A is g**-open.

4. g - T_0 Space**

Definition 4.1: A topological space (X, τ) is said to be a g**- T_0 space if for every pair of points $x \neq y$ in X either there exists g**-open set U such that $x \in U,\ y \notin U$ or $y \in U,\ x \notin U$.

Example 4.2: Let (X, τ) be an indiscrete topological space with at least two points. Here all subsets are g-closed, only φ and X are g*-closed and all subsets are g**-closed. This space is g**-T_0.

Example 4.3: Let X = {a,b,c,d}, τ = {φ, {a}, X}.Then G**O(X) = {φ, {a}, X}.This space is not g**-T_0.

Theorem 4.4: Every T_0 space is g**-T_0 space but not conversely.

Proof is obvious since every open set is g**-open.

Example 4.5: The space in example (4.2) is g**-T_0 but not T_0.

Theorem 4.6: Let (X, τ) be a g**-multiplicative topological space. Then X is g**-T_0 space if and only if the g**-closures of distinct points are distinct.

Proof: Let (X, τ) be a g**-T_0 space. Let x and y be two distinct points in X. Then there exists g**-open set U such that $x \in U$ and $y \in X - U$. Since $X - U$ is g**-closed, $g^{**}cl(\{y\}) \subseteq X - U$. $\therefore g^{**}cl(\{x\}) \neq g^{**}cl(\{y\})$. Conversely let $g^{**}cl(\{x\}) \neq g^{**}cl(\{y\})$ whenever $x \neq y$. Then there exists $z \in g^{**}cl(\{x\})$ and $z \notin g^{**}cl(\{y\})$. Suppose $x \in g^{**}cl(\{y\})$, then $g^{**}cl(\{x\}) \subseteq g^{**}cl(g^{**}cl(\{y\}) = g^{**}cl(\{y\})$. Therefore $z \in g^{**}cl(\{y\})$, which is not true. Hence $x \notin g^{**}cl(\{y\})$. Since X is g**-multiplicative, $g^{**}cl(\{y\})$ is g**-closed. Therefore $U = X - g^{**}cl(\{y\})$ is g**-open, $x \in U$ and $y \notin U$. Therefore (X, τ) is a g**-T_0 space.

Definition 4.7: Let (X, τ) and (Y, σ) be two topological spaces and $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be g**-resolute if $f(U)$ is g**-open in Y whenever U is g**-open in X.

Theorem 4.8: Let (X,τ) and (Y,σ) be two topological spaces and $f:(X,\tau)\to(Y,\sigma)$ be a function. Then (1) f is one to one, g**-continuous and Y is a T_0 space $\Rightarrow$ X is a g**- T_0 space.

(2) f is one to one, g**-irresolute and Y is a g**-T_0 space $\Rightarrow$ X is a g**- T_0 space.

(3) f is one to one, continuous and Y is a T_0 space $\Rightarrow$ X is a g**- T_0 space.

(4) f is one to one, onto, g**-open and X is a T_0 space $\Rightarrow$ Y is a g**- T_0 space.

(5) f is one to one, onto, g**-resolute and X is a g**-T_0 space $\Rightarrow$ Y is a g**- T_0 space.

Proof: (1) Let x, y be two distinct points in X. Then $f(x)$ and $f(y)$ are distinct points in Y. Then there exists open set U in Y such that $f(x)\in U$ and $f(y)\notin U$ or $f(y)\in U$ and $f(x)\notin U$. Then $f^{-1}(U)$ is a g**-open set in X such that $x\in f^{-1}(U)$ and $y\notin f^{-1}(U)$ or $y\in f^{-1}(U)$ and $x\notin f^{-1}(U)$. Therefore X is a g**- T_0 space.

Proof for (2), (3), (4) and (5) are similar to (1).

The property of being g**-T_0 space is preserved under one to one, onto and g**-resolute mapping.

5. g**-T_0 modulo an Ideal

Definition 5.1: An ideal topological space (X,τ,I) is said to be g**-T_0 modulo *I* if for every pair of points $x\neq y$ in X there exists g**-open set U such that $x\in U, U\cap\{y\}\in I$ or $y\in U, U\cap\{x\}\in I$.

Example 5.2: Any ideal topological space (X,τ,I) with $I=\wp(x)$ is a g**-T_0 modulo *I* space.

Example 5.3: Let X = {a, b, c, d}, τ = {φ, {a}, X} and $I=\varphi$. In this space G**O(X) = {φ, {a}, X}. (X,τ,I) is not a g**-T_0 modulo *I* space.

Theorem 5.4: Every g**-T_0 space is g**-T_0 modulo *I* space for every ideal *I*.

Proof is obvious since $\varphi\in I$.

Remark 5.5: The converse of the above theorem need not be true as seen in the following example.

Example 5.6: Let X = {a, b, c, d}, τ = {φ, {a}, X} and $I = \wp(X)$. Then (X, τ, I) is g**-T_0 modulo *I* but not g**-T_0.

Remark 5.7: If, $I = \{\varphi\}$ both the concepts "g**-T_0" and "g**-T_0 modulo *I*" coincide.

Theorem 5.8: Let (X, τ, I) be g**-T_0 modulo *I* and *J* an ideal in X with $I \subseteq J$ then (X, τ, J) is a g**-T_0 modulo *J* space.

Proof is obvious.

Theorem 5.9: Let $f : (X, \tau, I) \to (Y, \sigma, f(I))$ be a bijection then,

(1) f is g**-resolute and (X, τ, I) is g**-T_0 modulo $I \Rightarrow (Y, \sigma, f(I))$ is g**-T_0 modulo $f(I)$.
(2) f is g**-open and (X, τ, I) is $T_0 \Rightarrow (Y, \sigma, f(I))$ is g**-T_0 modulo $f(I)$.
(3) f is an open mapping and (X, τ, I) is $T_0 \Rightarrow (Y, \sigma, f(I))$ is g**-T_0 modulo $f(I)$.
(4) f is g**-continuous and Y is T_0 modulo $f(I) \Rightarrow$ X is g**-T_0 modulo I.
(5) f is g**-irresolute and Y is T_0 modulo $f(I) \Rightarrow$ X is g**-T_0 modulo I.
(6) f is continuous and Y is T_0 modulo $f(I) \Rightarrow$ X is g**-T_0 modulo I.

Proof: (1) : Note that{ f (I) / I∈ I }is an ideal in Y. Let $y_1 \neq y_2 \in Y$. Since f is onto, there exists $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ Since X is g**-T_0 modulo *I* there exists g**-open set U such that $x_1 \in U, U \cap \{x_2\} \in I$ or $x_2 \in U, U \cap \{x_1\} \in I$. Since f is g**-resolute $f(U)$ is g**-open in Y and $y_1 \in f(U), f(U) \cap \{y_2\} \in f(I)$ or $y_2 \in f(U), f(U) \cap \{y_1\} \in f(I)$. Therefore $(Y, \sigma, f(I))$ is g**-T_0 modulo $f(I)$.

Proof of (2),(3), (4), (5) and (6) are similar to (1).

6. g**- T_1 space

Definition 6.1: A topological space (X, τ) is said to be a g**- T_1 space if for every pair of points $x \neq y$ in X there exists g**-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Example 6.2: The space in example (4.2) is g**- T_1.

Example 6.3: The space in example (4.3) is not g**- T_1.

Theorem 6.4: Every T_1 space is g**-T_1 space but not conversely.

Proof is obvious since every open set is g**-open.

Example 6.5: The space in example (4.2) is g**-T_0 but not T_0.

Theorem 6.6: A topological space (X,τ) is said to be a g**- T_1 space if and only if every singleton set is g**-closed.

Proof: Necessity: Let (X,τ) be a g**- T_1 space and $x_0 \in X$. Let $x \neq x_0$ be an arbitrary element in X. There exists g**-open sets U and V such that $x \in U, x_0 \notin U$ and $x_0 \in V, x \notin V$. Now U is a g**-open set containing x not intersecting $\{x_0\}$. Therefore x is not a g**-limit point of $\{x_0\}$. $\therefore \{x_0\}$ is g**-closed.(by theorem (3.6).

Sufficiency: Let every singleton set in X be g**-closed. If x and y are distinct points in X then $U = X - \{y\}$ and $V = X - \{x\}$ are g**-open sets in X such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Therefore (X,τ) is a g**-T_1 space.

Theorem 6.7: If (X,τ) is a g**-T_1 space then every finite subset of X is g**-T_1.

Proof: Let A be a finite subset then $A = \bigcup_{x \in A} \{x\}$ is a finite union of g**-closed sets and hence it is g**-closed.

Theorem 6.8: In a topological space (X,τ) the following statements are equivalent:

(1) (X,τ) is a g**-T_1 space.
(2) Every singleton set of (X,τ) is g**-closed.
(3) Every finite subset of X is g**-closed.
(4) The intersection of g**neighbourhoods of an arbitrary point of X is singleton.

Proof: The proof for $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follows from the definitions and by theorems (6.4) and (6.5)

$(1) \Rightarrow (4)$: Let N be the intersection of g**neighbourhoods of x in X. Let $y \neq x$ be a point in X.

Since (X,τ) is g**-T_1 there exists a g**-open set U such that $x \in U, y \notin U$. $\therefore y \notin N$. $\therefore N = \{x\}$.

(4)$\Rightarrow$(1): Let x and y be two distinct points in X and N be the intersection of all g**-open neighbourhoods of X. Then $N=\{x\}. \therefore y \notin N.$ Hence there exists at least one g**-open set U containing x and not containing $y.$ Similarly we can get a g**-open set V containing y and not containing $x.$ Therefore (X,τ) is a g**-T_1 space.

Remark 6.9:[3] Arbitrary union of g**-closed sets need not be g**-closed as seen in the following example..

Example 6.10: Consider R with cofinite topology. In this space G**C(X) = {φ, X, all finite subsets} Let A_n= {-n, -(n-1), ……………….n-1, n} then A_n's are g**-closed, but $\cup A_n$ = Z is not g**-closed.

Definition 6.11: The topological space (X, τ) is said to be g**-additive if arbitrary union of g**-closed sets is g**-closed. Equivalently arbitrary intersection of g**-open sets is g**-open.

Example 6.12: The space in example (4.2) is g**-additive.

Example 6.13: The space in example (6.11) (X, τ) is not g**-additive.

Definition 6.14: A space (X, τ) is said to be g**-discrete if every subset is g**-open. Equivalently every subset is g**-closed.

Example 6.15: All discrete topological spaces and all indiscrete topological spaces are g**-discrete.

Example 6.16: In example (6.11), is not g**-discrete.

Theorem 6.17: Every finite g**-T_1 space is a g**-discrete space.

Proof: Let (X, τ) be a finite g**-T_1 space and let A be a subset of X. Since A is finite it is g**-closed. Therefore (X, τ) is g**-discrete.

Theorem 6.18: Let (X, τ) be g**-additive and g**- T_1 space. Then (X, τ) is a g**-discrete space.

Proof: Let A be a subset of X. Then $A=\bigcup_{x\in A}\{x\}$ and each $\{x\}$ is g**-closed. Since X is g**-additive. A is g**-closed.$\therefore (X,\tau)$ is g**-discrete.

Theorem 6.19: Let (X,τ) be a g**-T_1 space and A be a subset of X. Then the following statements are equivalent.

(1) $x \in X$ is a g**-limit point of A.

(2) Every g**-open set containing x contains infinitely many points of A.

Proof: $(2) \Rightarrow (1)$ is obvious.

$(1) \Rightarrow (2)$ Let x be a g**-limit point of A and U be a g**-open set containing x. Suppose $A \cap U$ is finite, let $A \cap U = (x_1, x_2, ...x_n\}$. Since x is a g**-limit point of A $U \cap (A - \{x\}) \neq \varphi$. Then $H = U \cap \{A - \{x\}\}$ is finite and hence it is g**-closed. $\therefore H^c$ is g**-open and so $(H^c \cap U)$ is g**-open set containing x. $(H^c \cap U) \cap (A - \{x\}) = H^c \cap (U \cap (A - \{x\}) = H^c \cap H = \varphi,$ which is a contradiction to (1). $\therefore A \cap U$ is infinite.

Theorem 6.20: A finite subset of g**-T_1 space has no g**limit point.

Proof follows from theorem (6.18).

Theorem 6.21: Let (X, τ) and (Y, σ) be two topological spaces and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then (1) f is one to one, g**-continuous and Y is a T_1 space $\Rightarrow$ X is a g**- T_1 space.

(2) f is one to one, g**-irresolute and Y is a g**-T_1 space $\Rightarrow$ X is a g**- T_1 space.

(3) f is one to one, continuous and Y is a T_1 space $\Rightarrow$ X is a g**- T_1 space.

(4) f is one to one, onto, g**-open and X is a T_1 space $\Rightarrow$ Y is a g**- T_1 space.

(5) f is one to one, onto, g**-resolute and X is a g**-T_1 space $\Rightarrow$ Y is a g**- T_1 space.

Proof: (1) Let x, y be two distinct points in X. Then $f(x)$ and $f(y)$ are distinct points in Y. Then there exists open sets U in Y such that $f(x) \in U$, $f(y) \notin U$ and $f(y) \in U$, $f(x) \notin U$. Then $f^{-1}(U)$ is a g**-open set in X such that $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$ and $y \in f^{-1}(U)$, $x \notin f^{-1}(U)$. Therefore X is a g**- T_1 space.

Proof for (2), (3), (4) and (5) are similar to (1).

The property of being g**-T_1 space is preserved under one to one, onto and g**-resolute mapping.

7. g-T_1modulo an ideal**

Definition 7.1: An ideal topological space (X,τ,I) is said to be g**-T_1 modulo *I* if for every pair of points $x \neq y$ in X there exists g**-open sets U and V such that $x \in U$, $y \in V, U \cap \{y\} \in I, V \cap \{x\} \in I$.

Example 7.2: Any ideal topological space (X,τ,I) with $I = \wp(x)$ is a g**-T_1 modulo *I* space.

Example 7.3: (X,τ,I) in example (5.3) is not g**-T_1 modulo *I* space where $x_0 \in X$.

Theorem 7.4: Every g**-T_1 space is g**-T_1 modulo *I* space for every ideal *I*.

Proof is obvious since $\varphi \in I$.

Remark7.5: If $I = \{\varphi\}$, both concepts "g**-T_1" and "g**-T_1 modulo *I*"coincide. .

Theorem 7.6: Let (X,τ,I) be g**-T_1 modulo *I* and J an ideal in X with $I \subseteq J$. Then (X,τ,J) is a g**-T_1 modulo J .

Proof is obvious.

Theorem 7.7: Every ideal topological space which is g**-T_1 modulo *I* is g**-T_0 modulo *I*.

Proof follows from the definitions.

Remark 7.8: The converse of the above theorem need not be true as seen in the following example.

Example 7.9: Let $X = \{a,b,c,d\}, \tau = \{\varphi, X, \{a\}\}, I = \{\varphi,\{c\},\{b\},\{b,c\}\}$.Then G**O(X) = {φ, X, {a}}. Then (X,τ,I) is g**-T_0 modulo *I* but not g**-T_1 modulo *I*.

Theorem 7.10: Let $f:(X,\tau,I) \to (Y,\sigma, f(I))$ be a bijection then,

(1) f is g**-resolute and (X,τ,I) is g**-T_1 modulo $I \Rightarrow (Y,\sigma, f(I))$ is g**-T_1 modulo $f(I)$.
(2) f is g**-open and (X,τ,I) is T_1 modulo $I \Rightarrow (Y,\sigma, f(I))$ is g**-T_1 modulo $f(I)$.
(3) f is an open mapping and (X,τ,I) is T_1 modulo $I \Rightarrow (Y,\sigma, f(I))$ is g**-T_1 modulo $f(I)$.

Proof follows from definitions.

8. g- T_2 space**

Definition 8.1: A topological space (X,τ) is said to be a g**- T_2 space if for every pair of distinct points x, y in X there exists disjoint g**-open sets U and V in X such that $x \in U$ and $y \in V$.

Example 8.2: The space given in example (4.2) is g**-T_2.

Example 8.3: An infinite set with cofinite topology is not a g**-T_2 space.

Theorem 8.4: Every T_2 space is g**-T_2 space but not conversely.

Proof is obvious since every open set is g**-open.

Example 8.5: The space given in example (4.2) is g**-T_2 but not T_2.

Theorem 8.6: Every g**-T_2 space is g**-T_1 space but not conversely.

Proof is obvious from the definitions.

Example 8.7: The space in example (8.3) is g**-T_1but not g**-T_2.

Theorem 8.8: Let (X,τ) and (Y,σ) be two topological spaces and f and g be g**-continuous functions from X to Y. If Y is a T_2 space then $A = \{x / f(x) = g(x)\}$ is g**-closed in X.

Proof: If $x_0 \in X - A$ then $f(x_0) = g(x_0)$. Since Y is a T_2 space, there exists open sets U and V such that $U \cap V = \varphi, f(x_0) \in U$ and $g(x_0) \in V$. Then $x_0 \in f^{-1}(U) \cap g^{-1}(V) = N$ which is g**-open in X. Hence N is a g**neighbourhood of x_0 contained in $X - A$ which proves $X - A$ is g**-open.

Theorem 8.9: Let (X,τ) and (Y,σ) be two topological spaces and f and g be g**-irresolute functions from X to Y. If Y is a g**-T_2 space then $A = \{x / f(x) = g(x)\}$ is g**-closed in X.

Proof is similar to the above theorem.

Definition 8.10: We say a sequence $\{x_n\}$ in X is g**-convergent to x in X (briefly $x_n \xrightarrow{g^{**}} x$ if corresponding to every g**neighbourhood U of x there exists a positive integer N such that $x_n \in U$, for all $n \geq N$.

Theorem 8.11: If (X,τ) is a g**-T_2 space then a sequence of points of X, g**-converges to atmost one point of X.

Proof: Suppose that $x_n \xrightarrow{g^{**}} x$ and $x_n \xrightarrow{g^{**}} y$ where x and y are two distinct points in X. Since X is a g**-T_2 space, there exists disjoint g**-open sets U and V such that $x \in U$ and $y \in V$. Since $x_n \xrightarrow{g^{**}} x$ there exists N such that $x_n \in U$, for all $n \geq N$. Then V can contain only finitely many points of the sequence $\{x_n\}$, x_n does not g**-converge to y.

Theorem 8.12: Every g**- discrete topological space, every discrete space and every indiscrete space is g**-T_2 , g**-T_1 and g**-T_0 .

Proof: All discrete spaces and indiscrete spaces are g**-discrete. In a g**-discrete topological space all subsets are g**-open. Let x and y be two distinct points in X. Then $U = \{x\}$ and $V = \{y\}$ are disjoint g**-open sets such that $x \in U$ and $y \in V$ and $U \cap V = \varphi$. Therefore (X,τ) is a g**-T_2 space and hence is g**-T_1 and g**-T_0 .

Definition 8.13: If $A: X \to X$ is a function then define Fix(A) = $\{ x \in X \,/\, Ax = x\}$.

Theorem 8.14: Let (X,τ) be a g**-T_2 space and f be an irresolute function of X into itself then Fix(f) is g**-closed.

Proof: It is enough to prove that $X - A$ is g**-open. Suppose $X - A$ is empty then it is g**-open. Let $X - A \neq \varphi$, then there exists $x_0 \in X - A$. $\therefore f(x_0) \neq x_0$. Then there exists disjoint g**-open sets U and V such that $x_0 \in U$ and $f(x_0) \in V$. Then $x_0 \in f^{-1}(V)$ which is g**-open. $\therefore U \cap f^{-1}(V)$ is g**-open set containing x_0. If $x \in U \cap f^{-1}(V)$ then $x \in U$ and $f(x) \in V$. $\therefore x \neq f(x)$ which implies $x \notin A$. $\therefore U \cap f^{-1}(V) \subseteq X - A$. $\therefore X - A$ is g**-open.

Theorem 8.15: Let (X,τ) be a T_2 space and f be a continuous function of X into itself then Fix (f) is g**-closed.

Proof is similar to the above theorem.

9. g-T_2 modulo an ideal**

Definition 9.1: An ideal topological space (X,τ,I) is said to be g**-T_2 modulo *I* if for every pair of distinct points x,y in X there exists g**-open sets U and V such that $x\in U-V, y\in V-U$ and $U\cap V\in I$.

Example 9.2: An indiscrete topological space (X,τ,I) is g**-T_2 modulo *I* for any ideal *I*.

Example 9.3: Let X be an infinite set, τ the cofinite topology and $I=\{\varphi\}$. Inthis space G**O(X) = {φ, X, A / A^c}. It is impossible to find two disjoint g**-open sets. Therefore this space is not g**-T_2 modulo *I*.

Theorem 9.4: Every g**-T_2 space is g**-T_2 modulo *I* but not conversely.

Example 9.5: In example (9.3) if $I=\wp(X)$ then the space is not g**-T_2 but it is g**-T_2 modulo *I*.

For, if x,y are distinct points in X then $U=X-\{x\}$, $V=X-\{y\}$ are g**-open sets such that $x\in V-U, y\in U-V$ and $U\cap V\in I$.

Note: When $I=\{\varphi\}$ the concepts "g**-T_2" and "g**-T_2 modulo *I*" coincide.

Theorem 9.6: Let (X,τ,I) be g**-T_2 modulo *I* and J an ideal in X with $I\subseteq J$. Then (X,τ,J) is a g**-T_2 modulo J.

Proof is obvious.

Theorem 9.7: Every ideal topological space which is g**-T_2 modulo *I* is g**-T_1 modulo *I*.

Proof follows from the definitions.

Remark 9.8: The converse of the above theorem need not be true as seen in the following example.

Example 9.9: Let $X=\{a,b,c\}, \tau=\{\varphi,X,\{a\},\{a,c\}\}, I=\wp(X)$. Then G**O(X) = {φ, X, {b}, {a, b}, {b, c}}. Then (X,τ,I) is g**-T_1 modulo *I* but not g**-T_2 modulo *I*.

Theorem 9.10: Let $f:(X,\tau,I)\to(Y,\sigma,f(I))$ be a bijection then,

(1) f is g**-resolute and (X,τ,I) is g**-T_2 modulo $I \Rightarrow (Y,\sigma,f(I))$ is g**-T_2 modulo $f(I)$.

(2) f is g**-open and (X,τ,I) is T_2 modulo $I \Rightarrow (Y,\sigma, f(I))$ is g**-T_2 modulo $f(I)$.

(3) f is an open mapping and (X,τ,I) is T_2 modulo $I \Rightarrow (Y,\sigma, f(I))$ is g**-T_2 modulo $f(I)$.

10. g**-regular spaces and g**-T_3 spaces

Definition10.1: A topological space (X, τ) is said to be g**-regular if and only if for every closed subset F of X and for each point $x \notin F$ there exists two disjoint g**-open sets G and H such that $x \in G$ and $F \subseteq H$.

Example 10.2: Any indiscrete space (X, τ) is g**-regular.

Example 10.3: The space (X, τ) in example 9.3) is not g**-regular.

Theorem 10.4: Every regular space is g**-regular.

Proof: Obvious, since every open set is g**-open.

Definition 10.5: A topological space (X, τ) is said to be g**-T_3 space if it is g**-regular and g**-T_1.

Example 10.6: The space in example (9.3) is g**-T_1 but not g**-regular and hence not g**-T_3.

Example 10.7: The space in (10.2) is g**-regular and g**-T_1 and so g**-T_3.

11. g**-regular spaces and g**-T_3 spaces

Definition11.1: A topological space (X, τ) is said to be g**-regular modulo *I* if for every closed subset F of X and for each point $x \notin F$ there exists two g**-open sets G and H such that $x \in G-H, F \subseteq H-G$ and $G \cap H \in I$.

Example 11.2: Any indiscrete space (X, τ) is g**-regular modulo *I* for any ideal *I*.

Example 11.3: (X,τ,I) in example (9.3) is not g**-regular modulo *I*.

Theorem 11.4: Every g**-regular space is g**-regular modulo *I* for any ideal but not conversely.

Example 11.5: In example (9.3) if $I = \wp(X)$ then (X, τ, I) is g**-regular modulo I but not g**-regular.

Note: If $I = \{\varphi)$ then both g**-regular and g**-regular modulo I coincide.

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