b-separation axioms and τ^{*b} –Topology

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Abstract: In this paper, the characterization of basic open sets and subbasic open sets are introduced and discussed in τ^{*b} - topology

Keywords: Basic open sets and sub-basic open sets in τ^{*b} Topology,b- Compactness modulo an ideal,b-countably compact modulo $\mathcal{J}$, b-Lindelof modulo an ideal,bT_0 modulo an ideal, bT_1 modulo an ideal , bT_2 modulo an ideal, bT_3 modulo an ideal, b-normal modulo an ideal

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1. Introduction: Ideal topological spaces have been first introduced by K. Kuratowski**[4]** in 1930. R. Vaidyanathaswamy,**[8]** introduced local function in 1945 and defined a topology τ*. Andrijevic**[2]**introduced a new class of b-open sets in a topological space in 1996. Pauline Mary Helen, et.al **[6]**, ,introduced b-local function and obtained τ^{*b} -topology in *b-finitely additive space .In this paper, the characterization of basic open sets and subbasic open sets are introduced and discussed in τ^{*b} - topology.

2. Preliminaries and definitions

Definition 2.1:[3]An ideal $\mathcal{J}$ on a nonempty set X is a collection of subsets of X which satisfies the following properties:(i)A$\in\mathcal{J}$ and B$\subseteq$A implies B$\in\mathcal{J}$(ii)A$\in\mathcal{J}$ and B$\in\mathcal{J}$ implies A$\cup$B$\in\mathcal{J}$

Definition 2.2:[3]A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by (X, τ,$\mathcal{I}$). Let Y be a subset of X. Then, $\mathcal{I}_Y$ = {I∩Y/I ∈$\mathcal{I}$} is an ideal on Y and $\mathcal{I}_Y$ = {G∩Y/G ∈τ} is a topology on Y. By (Y, τ /Y,$\mathcal{I}_Y$) we denote the ideal topological subspace.

Definition 2.3:[9]Let $\mathcal{P}$(X) be the power set of X , then a set operator ( )*:$\mathcal{P}$(X)→ $\mathcal{P}$(X), called the local function of A with respect to τ and $\mathcal{I}$ is defined as follows: For A⊆X, A*($\mathcal{I}$, τ) ={x ∈ X/U∩A∉$\mathcal{I}$ for every open set U containing x}.

We simply write A* instead of A*($\mathcal{I}$, τ) in case there is no confusion. X* is often a proper subset of X.

Definition 2.4:[3]A Kuratowski closure operator cl*( ) for a topology τ^*($\mathcal{I}$, τ), called the τ *-topology is defined by cl*(A)= A∪ A*.τ^*($\mathcal{I}$) is finer than τ and β($\mathcal{I}$, τ) = {U-I/U∈ τ, I∈$\mathcal{I}$) is a basis for τ^*($\mathcal{I}$).

Definition 2.5:[3]A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be τ^* -closed or simply *-closed if A*⊆ A.

For a subset A of X, cl(A) denotes the closure of A in (X, τ). Similarly cl*(A) and int*(A) will represent the closure of A and the interior of A in (X,τ^*) respectively.

Definition 2.6:[2] A subset A of X is said to be b-open if A ⊆cl int (A)∪int cl(A).The complement of b- open set is called b- closed. The collection of all b-open sets and b-closed sets are denoted by BO(X, τ) and BC(X, τ).

Note 2.7:[2]

1.Arbitrary union of b-open sets is b-open.

2. Intersection of b-open sets need not b-open.Equivalently union of b-closed sets need not b-closed.

Definition 2.8:[6]Let (X, τ, $\mathcal{J}$) be an ideal topological space and A subset of X. Then$A^{*b}(\mathcal{J}, \tau)$={x ∈X/A∩U∉$\mathcal{J}$ for every U∈BO(X, x)} is called the b-local function of A with respect to $\mathcal{J}$ and τ, where BO(X, x) = {U∈BO(X)/ x ∈U}.cl^{*b}(A) is defined to be A∪A^{*b}.

Remark 2.9:[6]Let (X, τ, $\mathcal{J}$) be an ideal topological space and A, B subsets of X. Then, for the b-local function, the following properties hold:

1. If A⊂B, then $A^{*b} \subset B^{*b}$.
2. A^{*b}= bcl (A^{*b}) ⊂bcl(A) and A^{*b} is b-closed in X
3. $\left(A^{*b}\right)^{*b} \subseteq A^{*b}$.
4. $(A \cup B)^{*b} \subseteq A^{*b} \cup B^{*b}$.

In general $(A \cup B)^{*b} \neq A^{*b} \cup B^{*b}$.

cl^{*b}(A) satisfies the following properties.

1. cl^{*b}(φ)=φ
2. cl^{*b}(cl^{*b}(A))=cl^{*b}(A)
3. cl^{*b}(A∪B) ⊆cl^{*b}(A)∪cl^{*b}(B)
4. $A \subseteq cl^{*b}$(A)

Definition 2.10:[6]An ideal topological space (X, τ, $\mathcal{J}$) is said to be

(1) *b- finitely additive if$[\cup_{i=1}^{\infty} A_i]^{*b} = \cup_{i=1}^{\infty}(A_i)^{*b}$.

(2) *b-Countably additive if $[\cup_{i=1}^{\infty} A_i]^{*b} = \cup_{i=1}^{\infty}(A_i)^{*b}$.

(3) *b –additive if $(\cup_{\alpha} A_{\alpha})^{*b} = \cup_{\boldsymbol{\alpha}} (\boldsymbol{A_{\alpha}},)^{*b}$ for arbitrary collection$\{A_{\alpha}\}$.

Note 2.11: [6]In a *b- finitely additive space,

cl^{*b}(A∪B)= (A∪B) $\cup (A \cup B)^{*b}$= (A∪B) $\cup (A^{*b} \cup B^{*b})$= cl^{*b}(A) $\cup cl^{*b}$(B)

If (X, τ, $\mathcal{I}$) is *b-finitely additive, thencl^{*b}(A) satisfies the Kurtowski closure axiom. Therefore τ^{*b}= {A X/cl^{*b}(X-A)= X-A} is a topology on X.

Definition 2.12:[2]Let f:(X, τ)→(Y, σ) be a map. Then, f is said to be

(1) b-continuous if f^{-1} (A) is b-open whenever A is open in Y.

(2) b-open if f (A) is b-open whenever A is open in X.

(3) Strongly b-continuous if f^{-1} (A) is open in X whenever A is b-open in Y.

(4) b-irresolute if f^{-1} (A) is b-open in X whenever A is b-open in Y.

(5) b-resolute if f (A) is b-open whenever A is b-open in X.

Definition 2.13:[7]Let f:(X, τ)→(Y, σ) be a map. Then, f is said to be b-homeomorphism if f is a i)bijection ii) b-irresolute iii) b-resolute.

Definition 2.14:[7]Any property of X which is entirely expressed in terms of the topology of X yields,(via the b-homeomorphism f) the corresponding property for the space Y is called a b-topological property of X.

Definition 2.15:[4] A subset Y of an ideal topological space (X, τ, $\mathcal{I}$) is said to be b-compact if for every cover $\{G_\alpha / \alpha \in \Omega\}$ of X by b-open sets G_α in X, there exist a finite subset Ω_0 of Ω such that X = $\bigcup_{\alpha \in \Omega_0} G_\alpha$.

Definition 2.16:[7]A topological space (X, τ) is said to be b- Lindelof if for every b- open cover $\{U_\alpha\}_{\alpha\in\Omega}$ of X there exists a countable subset Ω_0 of Ω such that X -$\bigcup_{\alpha\in\Omega_0} U_\alpha$.

Definition 2.17:[8]An ideal topological space (X, τ, $\mathcal{I}$) is said to be Lindelof modulo $\mathcal{I}$ if for every open cover$\{U_\alpha\}_{\alpha\in\Omega}$ of X there exists a countable subset Ω_0 of Ω such that X – $\bigcup_{\alpha\in\Omega_0} U_\alpha \in \mathcal{I}$.

Definition 2.18:[7] A collection $\mathcal{B}$ of b-open sets is said to be a b- basis for X if for every b-open set U and x ∈U there exist B∈$\mathcal{B}$ such that x ∈B⊆ U. The members of $\mathcal{B}$ are called basic b-open sets.

Definition 2.19:[8] A topological space (X, τ) is said to be bT_0 if for any two points x ≠ y in X there exist b-open set G such that x ∈G, y ∉G or y∈ G, x ∉G.

Definition 2.20:[8]An ideal topological space (X, τ, $\mathcal{I}$) is said to be T_0modulo $\mathcal{I}$ if for any two points x ≠ y in X, there exist a open set G such that x ∈ G and G∩{y} ∈$\mathcal{I}$ or y ∈ G and G∩{x} ∈$\mathcal{I}$.

Definition 2.21:[7] A topological space (X, τ) is said to be bT_1 if for every two points x ≠ y in X there exist b-open sets U, V∈ τ such that x∈ U\V, y∈ V\U. Equivalently, every singleton set is b-closed.

Definition 2.22:[8]An ideal topological space (X, τ, $\mathcal{I}$) is said to be T_1modulo $\mathcal{I}$ if for any two points x ≠ y in X, there exist a open sets U, V∈ τ such that x∈ U\V, y∈ V\U and U∩{y} ∈$\mathcal{I}$ and V∩{x} ∈$\mathcal{I}$.

Definition 2.23:[7] A topological space (X, τ) is said to be bT_2 if for every two points x ≠ y in X there exist disjoint b-open sets U,V in X such that x∈ U, y∈ V.

Definition 2.24 :[8]An ideal topological space (X, τ, $\mathcal{I}$) is said to be T_2modulo $\mathcal{I}$ if for every points x≠ y in X, there exist a open sets U, V in X such that x∈ U\V, y∈ V\U and U∩V ∈$\mathcal{I}$.

Definition 2.25:[7]A topological space (X, τ) is said to be a bT_3space or b-regular space,if

(1)X is b T_1space and

(2)For any b-closed subset F of X and every point x ∉F there exist disjoint b-open sets G, H in X such that x ∈H, F⊆G.

Definition 2.26:[8]An ideal topological space (X, τ, $\mathcal{I}$) is said to be a T_3modulo $\mathcal{I}$ space or regular modulo an ideal space. If

(1) X is T_1modulo $\mathcal{I}$ and

(2) For any semi closed subset F of X and every point x ∉F there exist b-open sets G, H such that x ∈H\G, F⊆G-H and G∩H ∈$\mathcal{I}$.

Definition 2.27:[7]A topological space (X, τ) is said to be b-normal if

(1) X is semi T_1space.

(2) For any two disjoint semi closed sets G, H in X, there exist disjoint b-open sets

U, V such that G ⊆U, H ⊆V.

Definition 2.28:[8]An ideal topological space (X, τ, $\mathcal{I}$) is said to be normal modulo $\mathcal{I}$ if

(1) X is T_1modulo $\mathcal{I}$.

(2) For any two disjoint closed sets G, H in X, there exists open sets U, V such that G ⊆U\V, H ⊆V\U and G∩H ∈$\mathcal{I}$.

3.Basic open sets and sub-basic open sets in τ^{*b} Topology

Definition 3.1[7]: A topological space (X, τ) is said to be

(1) finitely b-additive if finite union of b-closed sets is b-closed.

(2) Countablyb-additive if finite union of b-closed sets is b-closed.

(3) b-additive if arbitrary union of b-closed sets is b-closed.

Remark 3.2[7] :b-additive ⇒Countably b-additive ⇒ Finitely b-additive.

Example 3.3: Let (X, τ) be an infinite cofinite topological space. Then,τ={φ, X, A/A^c isfinite}, BO(X)= {φ, X, A/A is infinite}. In this space a set A is b-closed ⇔ A^c is infinite.

This space is not finitely b-additive and hence it is not countably b-additive andb-additive.

Theorem 3.4: If (X, τ, $\mathcal{I}$) is finitely *b-additive and finitely b-additive space then $\mathcal{B}$ = {V-I/V∈ BO(X), I∈$\mathcal{I}$} is a basis for the topology τ^{*b}.

Proof: If (X, τ, $\mathcal{I}$) is *b-finitely additive then τ^{*b} is a topology. U ∈τ^{*b}⇔ X-U is τ^{*b}–closed ⇔$(X-U)^{*b}$⊆ X-U ⇔U ⊆ X- $(X-U)^{*b}$∴ x ∈ U ⇒ x ∉ $(X-U)^{*b}$⇒there exist V ∈ BO(X, x) such that V∩(X-U) ∈$\mathcal{I}$. Let I= V ∩ (X-U) then x ∉ I which implies x ∈ V-I ⊆ U.

It is enough to prove: Intersection of two members of $\mathcal{B}$ is again in $\mathcal{B}$.Let x $\in (V_1 - I_1) \cap (V_2 - I_2)$ where V_1, $V_2 \in$ BO(X) and $I_1, I_2 \in \mathcal{J}$.Then, $(V_1 - I_1) \cap (V_2 - I_2) = V_1 \cap I_1^c \cap V_2 \cap I_2^c$ $= (V_1 \cap V_2) \cap (I_1 \cup I_2)^c = (V_1 \cap V_2) - (I_1 \cup I_2) \in \mathcal{B}$, since $V_1 \cap V_2$ is b-open and $I_1 \cup I_2 \in \mathcal{J}$.$\therefore \mathcal{B}$ is a basis for τ^{*b}.

Note 3.5: If (X, τ, $\mathcal{J}$) is finitely *b-additive and not finitely b-additive then $\mathcal{B}$ is only a sub basis forτ^{*b}.

Example 3.6: Let(X, τ) be an infinite cofinite topological space and $\mathcal{J} = \mathcal{P}$(X).Then $A^{*b} = \phi$ for any subset A and BO(X) = { ϕ,X,all infinite subsets}.Therefore (X, τ) is not finitely b-additive.But $cl^{*b}(A) = A \cup A^{*b} = A$ for all $A \subseteq X$. Therefore (X, τ, $\mathcal{J}$) is finitely *b additive. So in this space B need not be a subbasis for τ^{*b}

Theorem 3.7: In (X, τ) a set is b-open if and only if it is union of b-basic open sets.

Proof: Let U be b-open. For every x$\in$ U there exist $B_x \in \mathcal{B}$ such that x $\in B_x \subseteq$U.ThereforeU = $\bigcup_{x \in U} B_x$.Conversely ,union of b-basic open sets is b-open,since every b -basic open set is b-open and union of b-open sets is b-open.

4.b-COMPACTNESS MODULO AN IDEAL

Definition4.1: An ideal topological space (X, τ, $\mathcal{J}$) is said to be b-compact modulo $\mathcal{J}$ if for every b-open cover $\{G_\alpha / \alpha \in \Omega\}$ of X there exists a finite subset Ω_0 of Ω such that $X - \bigcup_{\alpha \in \Omega_0} G_\alpha \in \mathcal{J}$.

Remark 4.2:

(1) Every finite ideal topological space (X, τ, $\mathcal{J}$) is b-compact modulo $\mathcal{J}$.

(2) Every b-compact modulo $\mathcal{J}$ space is compact modulo $\mathcal{J}$, since every open set is b-open.

(3) Every b-compact space is b-compact modulo $\mathcal{I}$, for any ideal $\mathcal{I}$ since φ ∈$\mathcal{I}$ but not conversely as seen in the following example.

Example 4.3: Consider an infinite discrete space (X, τ) and an ideal $\mathcal{P}$(X). In this space BO(X)={all subsets}.{{x}/x∈ X} is a b-open cover which has no finite sub cover. ∴(X, τ) is not b-compact. On the other hand if $\{G_\alpha/\alpha \in \Omega\}$ is a b-open cover for X, and Ω_0 is any finite subset of Ω, then X – $\bigcup_{\alpha\in\Omega_0} G_\alpha$∈$\mathcal{P}$(X).∴(X, τ, $\mathcal{P}$(X)) is b-compact modulo $\mathcal{P}$(X).But (X, τ) is not b-compact

Theorem 4.4: Let (X, τ, $\mathcal{I}$) be a b-compact modulo $\mathcal{I}$ space. Then every b-closed subset of X is b-compact modulo $\mathcal{I}$.

Proof: Let A be a b-closed subset of X and $\{G_\alpha\}_{\alpha\in\Omega}$ be a cover for A by b-open sets in X.Then $\{\{G_\alpha\}_{\alpha\in\Omega}$,X-A} is a b-open cover for X. By the hypothesis there exist a finite sub cover such that X – {G_{α_1}∪ ….. ∪G_{α_n}∪ (X-A)} ∈$\mathcal{I}$.Then A – {G_{α_1}∪ ….. ∪G_{α_n}} ∈$\mathcal{I}$. ∴ A is b-compact modulo $\mathcal{I}$.

Theorem 4.7: If f : X→Y is a bijection then J is an ideal in Y, ⇔f^{-1}(J) is an ideal in X.

Proof: Obvious from the definition.

Theorem 4.8: Let f:(X, τ, $\mathcal{I}$)→(Y, σ, f($\mathcal{I}$)) be a bijection. Then

(1) X is b-compact modulo $\mathcal{I}$ and f is b-irresolute ⇒Y is b-compact modulo f($\mathcal{I}$).

(2) X is b-compact modulo $\mathcal{I}$ and f is b-continuous⇒Y is compact modulo f($\mathcal{I}$).

(3) X is compact modulo $\mathcal{I}$ and f is strongly b-continuous ⇒Y is b-compact modulo $\mathcal{I}$.

(4) Y is b-compact modulo f($\mathcal{I}$) and f is b-resolute ⇒X is b-compact modulo $\mathcal{I}$.

(5) Y is b-compact modulo f($\mathcal{I}$) and f is b-open⇒X iscompact modulo $\mathcal{I}$.

Proof: Let $\{G_\alpha\}_{\alpha\in\Omega}$ be a b-open cover for Y. Since f is b-irresolute, $f^{-1}(G_\alpha)$ is b-open in X for all $\alpha \in \Omega$.∴$\{f^{-1}(G_\alpha)\}_{\alpha\in\Omega}$is a b-open cover for X. Since X is b-compact modulo $\mathcal{I}$, there exists a finite subset Ω_0 of Ω such that X – $\bigcup_{\alpha\in\Omega_0} f^{-1}(G_\alpha)\in\mathcal{I}$.∴ Y-$\bigcup_{\alpha\in\Omega_0}(G_\alpha)\in$ f ($\mathcal{I}$).∴ Y is b-compact modulo f ($\mathcal{I}$).

Proof of (2) to (4) are similar.

Remark 4.9: From Theorem 4.8 (1) and (4) ,it follows that 'b-compact modulo $\mathcal{I}$' is a b-topological property.

Theorem 4.10: Let $\mathcal{I}_F$denote the ideal of all finite subsets of X. Then (X,τ) is b-compact if and only if (X, τ,$\mathcal{I}_F$) is b-compact modulo$\mathcal{I}_F$.

Proof: Let (X, τ) be b-compact.Then by the remark 4.2(3), (X, τ,$\mathcal{I}_F$) is b-compact modulo$\mathcal{I}_F$.

Conversely let (X, τ,$\mathcal{I}_F$) be b-compact modulo $\mathcal{I}_F$.Let $\{G_\alpha\}_{\alpha\in\Omega}$ be a b-open covering for X.

Then there exist a finite subset Ω_0 of Ω such that X – $\bigcup_{\alpha\in\Omega_0}(G_\alpha)\in\mathcal{I}_F$ .Let X-$\bigcup_{\alpha\in\Omega_0}(G_\alpha)$ = $\{x_1,x_2,\ldots\ldots,x_n\}$ and let $x_i\in G_{\alpha_i}$ for i=1,2,3,……,n.Then X = $\left\{\bigcup_{\alpha\in\Omega_0} G_\alpha\right\} \cup \{\bigcup_{i=1}^{n} G_{\alpha_i}\}$. ∴ (X, τ) is b-compact.

5. b-COUNTABLY COMPACT MODULO$\mathcal{I}$

Definition 5.1 : A subset A of a topological space (X, τ) is said to be b-countably compact if every countable b-open covering of A has a finite sub cover.

Definition 5.2: An ideal topological space (X, τ, $\mathcal{I}$) is said to be b-countablycompact modulo $\mathcal{I}$ if for every countable b-open cover $\{G_\alpha/\alpha \in \Omega\}$ of X, there exists a finite subset Ω_0 of Ω such that $X - \bigcup_{\alpha \in \Omega_0} G_\alpha \in \mathcal{I}$.

All the results from remark (4.2) to theorem (4.10)are true in the case when (X, τ, $\mathcal{I}$) is b-countably compact modulo $\mathcal{I}$.

Remark 5.3:

(1) b-compact modulo $\mathcal{I}$ implies b-countably compact modulo $\mathcal{I}$ for $\varphi \in \mathcal{I}$

(2) Every finite ideal topological space (X, τ, $\mathcal{I}$) is b- countably compact modulo $\mathcal{I}$.

(3) Every b- countably compact modulo $\mathcal{I}$ space is countably compact modulo $\mathcal{I}$, since every open set is b-open.

(4) Every b-countably compact space is b-countably compact modulo $\mathcal{I}$, for any ideal $\mathcal{I}$ since $\varphi \in \mathcal{I}$

Theorem 5.4: Let (X, τ, $\mathcal{I}$) be a b-countably compact modulo $\mathcal{I}$ space. Then every b-closed subset of X is b-countably compact modulo $\mathcal{I}$.

Proof: Proof is similar to the proof of theorem (4.4)

Theorem 5.5: If f : X→Y is a bijection then J is an ideal in Y, ⇔f^{-1}(J) is an ideal in X.

Proof: Obvious from the definition.

Theorem 5.6: Let f:(X, τ, $\mathcal{I}$)→(Y, σ, f($\mathcal{I}$)) be a bijection. Then

(1) X is b-countably compact modulo $\mathcal{I}$ and f is b-irresolute ⇒Y is b- countably compact modulo f($\mathcal{I}$).

(2) X is b- countably compact modulo $\mathcal{J}$ and f is b-continuous⇒Y is countably compact modulo f($\mathcal{J}$).

(3) X is countably compact modulo $\mathcal{J}$ and f is strongly b-continuous ⇒Y is b- countably compact modulo $\mathcal{J}$.

(4) Y is b- countably compact modulo f($\mathcal{J}$) and f is b-resolute ⇒X is b- countably compact modulo $\mathcal{J}$.

(5) Y is b- countably compact modulo f($\mathcal{J}$) and f is b-open⇒X is countably compact modulo $\mathcal{J}$.

Proof: Proof is similar to the proof of theorem (4.8).

Remark 5.7: From (1) and (4) of theorem 5.6, it follows that 'b-countably compact modulo $\mathcal{J}$' is a b- topological property.

Theorem 5.8: Let $\mathcal{J}_F$denote the ideal of all finite subsets of X. Then (X,τ) is b-countably compact if and only if (X, τ,$\mathcal{J}_F$) is b-countably compact modulo$\mathcal{J}_F$.

Proof: Proof is similar to theorem (4.10)

6 .b-LINDELOF MODULO AN IDEAL

Definition 6.1: An ideal topological space (X, τ, $\mathcal{J}$) is said to be b-Lindelof modulo $\mathcal{J}$ if for every b-open cover$\{U_\alpha\}_{\alpha\in\Omega}$ there exists a countable subset Ω_0 of Ω such that X $-\cup_{\alpha\in\Omega_0} U_\alpha \in\mathcal{J}$.

Remark 6.2:

(1) If (X, τ) is b-Lindelof then (X, τ, $\mathcal{J}$) is b-Lindelof modulo $\mathcal{J}$ for any ideal $\mathcal{J}$, since {φ}∈$\mathcal{J}$.

(2) If (X, τ, $\mathcal{J}$) is b-compact modulo $\mathcal{J}$ then (X, τ, $\mathcal{J}$) is b-Lindelof modulo $\mathcal{J}$. It follows from the definition.

(3) (X, τ) is b-Lindelof⇔(X, τ, φ) is b-Lindelof modulo {φ}. It follows from the definition.

(4) If (X, τ, $\mathcal{J}$) is b- Lindelof modulo $\mathcal{J}$ then (X, τ, $\mathcal{J}$) is Lindelof modulo $\mathcal{J}$ since every open set is b-open.

Theorem 6.3: Let$\mathcal{J}_c$ be the ideal of countable subsets of X. Then (X, τ) is b-Lindelof⇔

(X, τ, $\mathcal{J}_c$) is b-Lindelof modulo $\mathcal{J}_c$.

Proof: Proof is similar to theorem 4.10

Corollary 6.4: If (X, τ,$\mathcal{J}_c$) is b-compact modulo$\mathcal{J}_c$ then (X, τ) is b-Lindelof.

Proof: Follows from remark 6.2 (2) and theorem 6.5.

Theorem 6.5: A topological space (X, τ, $\mathcal{J}$) is b- Lindelof modulo $\mathcal{J}$ if and only if every b-basic open cover$\{B_\alpha\}$ has a countable sub collection $\{B_{\alpha_i}\}$ such that X $-\bigcup_{i=1}^{\infty} B_{\alpha_i} \in \mathcal{J}$.

Proof: Necessity: Obvious since b-basic open sets are b-open.

Sufficiency:Let$\{U_\alpha\}_{\alpha\in\Omega}$ be a b-open cover for X. By theorem (3.7), each U_α is a union of b- basic open sets $B_{\mathcal{B}}^{'}$s .So the collection of all such $B_{\mathcal{B}}^{'}$s is a b-basic open cover for X .By hypothesis, there exists a countable sub collection $\{B_{\mathcal{B}_i}$/i=1,2,…..} such that X- $\bigcup_{i=1}^{\infty} B_{\mathcal{B}_i} \in \mathcal{J}$.

For each set $B_{\mathcal{B}_i}$ in this countable collection, select a U_{α_i} which contains it. Then $\{U_{\alpha_i}\}_{i=1}^{\infty}$ is a countable sub cover of the collection $\{U_\alpha\}_{\alpha\in\Omega}$ and X $-\bigcup_{i=1}^{\infty} U_{\alpha_i} \subseteq$ X $-\bigcup_{i=1}^{\infty} B_{\mathcal{B}_i} \in \mathcal{J}$.Therefore (X, τ, $\mathcal{J}$) is b- Lindelof modulo $\mathcal{J}$.

Definition 6.6:A subset A of (X, τ, $\mathcal{J}$) is said to be b –Lindelof modulo $\mathcal{J}$ if every cover

$\{U_\alpha/\alpha \in \Omega\}$ of A by b-open sets in X has a countable sub cover $\{U_{\alpha_i}\}_{i=1}^{\infty}$ such that A-$\bigcup_{i=1}^{\infty} U_{\alpha_i} \in \mathcal{J}$.

Theorem 6.7: Let (X, τ, $\mathcal{J}$) be b- Lindelof modulo $\mathcal{J}$ space. Then every b-closed subset of X is b-Lindelof modulo $\mathcal{J}$.

Proof: Similar to proof of theorem (4.4)

Theorem 6.8: Let f:(X, τ, $\mathcal{J}$)→(Y, σ, f($\mathcal{J}$)) be a bijection. Then

1) X is b-Lindelof modulo $\mathcal{J}$ and f is b-irresolute ⇒Y is b-Lindelof modulo f($\mathcal{J}$).

2)X is b-Lindelof modulo $\mathcal{J}$ and f is b-continuous ⇒Y is Lindelof modulo f($\mathcal{J}$).

3)X is Lindelofmodulo $\mathcal{J}$ and f is strongly b-continuous ⇒Y is b-Lindelof modulo $\mathcal{J}$.

4)Y is b-Lindelofmodulo f($\mathcal{J}$) and f is b- resolute ⇒X is b-Lindelof modulo $\mathcal{J}$.

Proof: Similar to proof of theorem 4.8

Remark 6.9: From theorem 6.8(1) and (4) it follows that 'b-Lindelof modulo $\mathcal{J}$' is a b-topological property.

Theorem 6.10: Let (X, τ, $\mathcal{J}$) be finitely *b additive space. Then if (X,τ^{*b},$\mathcal{J}$) is Lindelof modulo $\mathcal{J}$ then (X, τ, $\mathcal{J}$) is b- Lindelof modulo $\mathcal{J}$. The converse is true if X is also finitely b-additive and $\mathcal{J}$is closed under countable union.

Proof: Necessity:Under the given hypothesis τ^{*b} is a topology. Let $\{U_\alpha\}_{\alpha\in\Omega}$ be a b-open cover for X. Since BO(X) $\subseteq\tau^{*b}$ and (X,τ^{*b},$\mathcal{J}$) is Lindelof modulo $\mathcal{J}$, there exist a countable subset Ω_0 of Ω such that X $-\bigcup_{\alpha\in\Omega_0} U_\alpha \in\mathcal{J}$.Hence (X, τ, $\mathcal{J}$) is b- Lindelof modulo $\mathcal{J}$.

Sufficiency:Since (X, τ) is finitely b- additive, {U-I/U ∈BO(X) and I∈$\mathcal{J}$} is a basis for τ^{*b}.Let $\{U_\alpha/\alpha \in \Omega\}$ be a cover for X by basic τ^{*b} -open sets.Then $U_\alpha = G_\alpha - I_\alpha$ where G_α∈BO(X) and $I_\alpha \in \mathcal{J}$.Here $\{G_\alpha\}$ is a b-open cover for X. Then there exists $\{G_{\alpha_i}$/i=1,2,…} such that X-$\bigcup_{i=1}^{\infty} G_{\alpha_i} \in \mathcal{J}$.∴ X- $\bigcup_{i=1}^{\infty} U_{\alpha_i}$= (X- $\bigcup_{i=1}^{\infty} G_{\alpha_i}$) ∪ ($\bigcup_{i=1}^{\infty} I_{\alpha_i}$) ∈$\mathcal{J}$.(Since $\mathcal{J}$ is closed under countable union.)∴(X,τ^{*b},$\mathcal{J}$) is Lindelof modulo $\mathcal{J}$

7.bT_0 MODULO AN IDEAL OR b-KOLMOGROV MODULO AN IDEAL

Definition 7.1: An ideal topological space (X, τ, $\mathcal{J}$) is said to be bT_0modulo $\mathcal{J}$ if for any two points x ≠ y in X, there exist G∈ BO(X) such that x ∈ G and G∩{y} ∈$\mathcal{J}$ or y ∈ G and G∩{x} ∈$\mathcal{J}$

Remark 7.2:

(1)EveryT_0space is bT_0, since every open set is b-open. Converse need not be true which follows from example (7.3)

(2)Every T_0 modulo $\mathcal{J}$ space is bT_0modulo $\mathcal{J}$, since every open set is b-open which follows from example (7.4)

(3)If (X, τ) is a bT_0 space then (X, τ, $\mathcal{J}$) is bT_0modulo $\mathcal{J}$ for any ideal $\mathcal{J}$ on X, since the open sets are b-open and φ ∈$\mathcal{J}$.The converse is not true .

Example 7.3: Let X={ a,b,c,d} $\tau = \{\phi, X, \{a,b\}\}$.

Then BO(X)={a},{b},{a,b},{a,c}{a,d},{b,c},ϕ, X.Therefore (X, τ) is not T_0 but bT_0

Example 7.4: Let X={ a,b,c,d} $\tau = \{\phi, X, \{a,b\}\}$ $\mathcal{J} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$is not T_0 modulo $\mathcal{J}$ but it is bT_0 modulo $\mathcal{J}$space

Theorem 7.5: Let (X, τ, $\mathcal{J}$) be bT_0modulo J and J an ideal in X with $\mathcal{J}\subseteq$ J. Then (X, τ, J) is

bT_0modulo J.

Proof: It is obvious.

Theorem 7.6: Let (X, τ, $\mathcal{J}$) be finitely *b-additive.Then (X, τ, $\mathcal{J}$) is bT_0modulo $\mathcal{J}\Rightarrow$ (X, τ^{*b}) is a T_0 space.The converse is true if (X, τ, $\mathcal{J}$) is finitely b-additive.

Proof: Let (X, τ, $\mathcal{J}$) be bT_0modulo $\mathcal{J}$ and x≠y be two points in X.Then there exist G ∈BO(X) such that x ∈ G and G∩{y} ∈$\mathcal{J}$or y ∈ G and G∩{x} ∈$\mathcal{J}$.Without loss of generality,assume that x ∈ G and G∩{y} ∈$\mathcal{J}$. If G∩{y} = φ,put G'=G.If G∩{y} ={y}, put G'=G-{y}.In both cases,G' is a sub basic open set inτ^{*b} topology.∴ $G'\in\tau^{*b}$, x∈G' and y∉G'.∴ (X, τ^{*b}) is a T_0 space.Conversely, Let (X, τ, $\mathcal{J}$) be finitely b-additive and let (X, τ^{*b}) be a T_0 space and x ≠ y be two points in X.Then $\mathcal{B}$ is a basis for τ^{*b}.Therefore there exist G'=G-I ∈τ^{*b} such that x∈G' and y∉G' or y∈G'andx∉$G'$where G ∈BO(X) and I ∈$\mathcal{J}$.w.l.g., let x∈G' and y∉G'.Then G∩{y} ={y} or {∅}. Suppose G∩{y} ={φ}, then G∩{y} ∈$\mathcal{J}$. Suppose G∩{y} ={y}, then y∈ I which implies {y}∈$\mathcal{J}$.Hence, G is a b-open set containing x and G∩{y} ∈$\mathcal{J}$.∴ (X, τ, $\mathcal{J}$) is bT_0modulo $\mathcal{J}$.

Theorem 7.7: Let (X, τ, $\mathcal{J}$) be finitely *b-additive T_0 space then (X, τ^{*b}) is a T_0 space.

Proof: It follows from the theorem (7.6) and (1) and (2) of remark (7.2).But the converse need not be true as seen from the following example.

Example 7.8: Consider an infinite discrete space (X, τ) and an ideal $\mathcal{P}$(X). Then τ^{*b}= {all subsets}.This space is *b-additive and (X, τ^{*b}) is T_0 but (X, τ) is not T_0.

Theorem 7.9: Let (X, τ, $\mathcal{J}$) be finitely *b-additive then (X, τ, $\mathcal{J}$) is bT_0modulo $\mathcal{J}\Rightarrow\tau^{*b}$ closure of distinct points are distinct. The converse is true if (X, τ, $\mathcal{J}$) is finitely b-additive.

Proof: Let $(X, \tau, \mathcal{I})$ be bT_0 modulo $\mathcal{I}$. By theorem (7.6), (X,τ^{*b}) is a T_0 space. If $x \neq y$ be two points in X, then there exist $U \in \tau^{*b}$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. w.l.g., assume that $x \in U$ and $y \notin U$. Then $y \in U^c$ which is τ^{*b} closed. $\therefore$ $x \notin cl^{*b}(y)$ which implies $cl^{*b}(x) \neq cl^{*b}(y)$. Conversely, let $x \neq y$ imply $cl^{*b}(x) \neq cl^{*b}(y)$. $G = [cl^{*b}(y)]^c$ is τ^{*b} open. Now, $G \in \tau^{*b}$, $x \in G$ and $y \notin G$ implies that (X,τ^{*b}) is a T_0 space. By theorem (7.6), $(X, \tau, \mathcal{I})$ is bT_0 modulo $\mathcal{I}$.

Theorem 7.10: Any b-homeomorphic image of bT_0 modulo $\mathcal{I}$ space is bT_0 modulo $f(\mathcal{I})$ where f is the corresponding b-homeomorphism.

Proof: $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, f(\mathcal{I}))$ is a b-homeomorphism. Let $y_1 \neq y_2$ be two points in Y. Then $y_1 = f(x_1)$, $y_2 = f(x_2)$ for some points $x_1 \neq x_2 \in X$. $\therefore$ There exist $G \in BO(X)$ such that $x_1 \in G$ and $G \cap \{x_2\} \in \mathcal{I}$ or $x_2 \in G$ and $G \cap \{x_1\} \in \mathcal{I}$. Then, $f(G)$ is b-open in Y. $\therefore$ $f(x_1) \in f(G)$, $f(G) \cap f(x_2) \in f(\mathcal{I})$ or $f(x_2) \in f(G)$, $f(G) \cap f(x_1) \in f(\mathcal{I})$. $\therefore$ $(Y, \sigma, f(\mathcal{I}))$ is bT_0 modulo $f(\mathcal{I})$.

Theorem 7.11: Let $f: (X, \tau, \mathcal{I}) \to (Y, \sigma, f(\mathcal{I}))$ be a bijection.

(1) f is b-open and X is T_0 modulo $\mathcal{I} \Rightarrow$ Y is bT_0 modulo $f(\mathcal{I})$

(2) f is b-resolute and X is bT_0 modulo $\mathcal{I} \Rightarrow$ Y is bT_0 modulo $f(\mathcal{I})$

(3) f is b-continuous and Y is T_0 modulo $f(\mathcal{I}) \Rightarrow$ X is bT_0 modulo $\mathcal{I}$.

(4) f is b-irresolute and Y is bT_0 modulo $f(\mathcal{I}) \Rightarrow$ X is bT_0 modulo $\mathcal{I}$.

(5) f is strongly b-continuous and Y is bT_0 modulo $f(\mathcal{I}) \Rightarrow$ X is T_0 modulo $\mathcal{I}$.

Proof: Obvious from the definition.

Note 7.11: The property of being 'bT_0 modulo $\mathcal{I}$' is a b-topological property as from (2) and (4) of theorem (7.11).

8.bT_1 MODULO AN IDEAL OR b-QUASI-SEPARATED MODULO AN IDEAL

Definition 8.1: An ideal topological space $(X,\tau,\mathcal{J})$ is said to be bT_1modulo $\mathcal{J}$ if for any two points $x \neq y \in X$, there exist b-open sets $U,V\in \tau$ such that $x \in U$, $y\in V$, $U\cap\{y\}\in\mathcal{J}$ and $V\cap\{x\} \in\mathcal{J}$.

Example 8.2: Any discrete ideal topological $(X,\tau,\mathcal{J})$ is bT_1modulo $\mathcal{J}$ for any ideal $\mathcal{J}$ since all subsets are b-open.

Example 8.3: Let X = {a,b,c,d} , $\tau = \{\varphi, X, \{a\}\}$.

Then BO(X) = {{a},{a,b},{a,c},{a,d},{a,b,c},{a,c,d},{a,b,d}, φ ,X}.Since {a} is not b-closed, (X, τ) is bT_0 but it is not bT_1.So $(X,\tau,\mathcal{J})$ is not bT_1 modulo $\mathcal{J}$ when $\mathcal{J} = \{\varphi\}$

Theorem 8.4: Every bT_1 space is bT_1 modulo $\mathcal{J}$ for any ideal $\mathcal{J}$.

Proof: Obvious since $\varphi \in \mathcal{J}$

Theorem 8.5: Let $(X, \tau, \mathcal{J})$ be bT_1 modulo J and J an ideal in X with $J \subseteq\mathcal{J}$. Then $(X, \tau, \mathcal{J})$ is bT_1 modulo$\mathcal{J}$.

Proof: Obvious

Theorem 8.9: Let $(X, \tau, \mathcal{J})$ be finitely *b-additive space. Then $(X, \tau, \mathcal{J})$ is bT_1 modulo $\mathcal{J}\Rightarrow$ (X, τ^{*b}) is a T_1 space. The converse is true if $(X, \tau, \mathcal{J})$ is a finitely b-additive space.

Proof: Similar to proof of theorem (7.6).

Theorem 8.10: Let $(X, \tau, \mathcal{J})$ be finitely *b-additive space. Then X is bT_1 modulo $\mathcal{J}\Rightarrow$ every singleton set is τ^{*b} closed. The converse is true if $(X, \tau, \mathcal{J})$ is also finitely b-additive.

Proof:X is a bT_1 modulo $\mathcal{J}\Rightarrow(X, \tau^{*b})$ is T_1 space. (By theorem (8.9))

$\Rightarrow$every singleton set is τ^{*b} closed.Conversely, every singleton set is τ^{*b} closed $\Rightarrow$(X, τ^{*b}) is a T_1 space,under the given hypothesis, (X, τ, $\mathcal{I}$) is bT_1modulo $\mathcal{I}$. (By theorem (8.9))

Theorem 8.11: Every space which is bT_1 modulo $\mathcal{I}$ is bT_0modulo $\mathcal{I}$.

Proof: The proof is obvious from the definition.Converse need not be true as seen from the following example.

Example 8.12: In example 8.3, (X,τ, φ) is not bT_1 modulo φ but it is bT_0 modulo φ

Theorem 8.13: Let (X, τ, $\mathcal{I}$) be finitely *b-additive, then (X,τ,$\mathcal{I}$) is bT_1 modulo $\mathcal{I}\Rightarrow$ every finite subset of X is τ^{*b}–closed. Converse is true if (X, τ, $\mathcal{I}$) is also finitely b-additive.

Proof: By theorem (8.4), in a bT_1 modulo $\mathcal{I}$ space, every singleton set is τ^{*b} closed and hence every finite subset is τ^{*b} closed since τ^{*b} is a topology.Conversely let every finite subset of X be τ^{*b}closed. Then in particular every singleton set is τ^{*b}closed. Therefore by theorem(8.10) (X, τ, $\mathcal{I}$) is bT_1 modulo $\mathcal{I}$.

Theorem 8.14: Let (X, τ, $\mathcal{I}$) be finitely *b-additive. Then (X, τ, $\mathcal{I}$) is bT_1 modulo $\mathcal{I}\Rightarrow\tau^{*b}$ contains the cofinite topology for X. The converse is true if (X, τ, $\mathcal{I}$) is also finitely b-additive.

Proof: Let (X, τ, $\mathcal{I}$) be bT_1 modulo $\mathcal{I}$ and A finite subset of X. By theorem (8.13), A is τ^{*b} closed.$\therefore A^c$ is τ^{*b} open.This shows that the complements of finite sets are τ^{*b}open and hence τ^{*b}contains the cofinite topology for X.Conversely, if τ^{*b}contains the cofinite topology for X

then for each $x \in X, X - \{x\}$ is τ^{*b}open.∴ $\{x\}$ is τ^{*b}closed.∴By theorem (8.13), $(X, \tau, \mathcal{I})$ is bT_1 modulo $\mathcal{I}$.

Theorem 8.15: Let $(X, \tau, \mathcal{I})$ be a finitely *b-additive, finite,bT_1 modulo $\mathcal{I}$ space ,then τ^{*b} is discrete topology.

Proof: By theorem (8.14) ,since every subset of X is τ^{*b}open and hence τ^{*b} is discrete topology.

Definition 8.16: Let$(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. We say x is a b-limit point of A modulo $\mathcal{I}$ if every τ^{*b} neighborhood of x contains atleast one point of A other than x.

Theorem 8.17: Let$(X, \tau, \mathcal{I})$ be finitely *b-additive, bT_1 modulo $\mathcal{I}$ space .Then x is a b-limit point of A modulo $\mathcal{I}$⇔ every τ^{*b}neighborhood of x contains infinitely many points of A.

Proof: If every τ^{*b} neighborhood of x contains infinitely many points of A then it contains atleast one point of A other than x.∴ x is a b- limit point of A modulo $\mathcal{I}$.Conversely, let x be a b-limit point of A modulo $\mathcal{I}$.Suppose that there exist a τ^{*b}neighborhood U of x which contains only finitely many points of A, then $U \cap \{A-\{x\}\}$ is finite .Let $U \cap \{A-\{x\}\} = \{x_1, x_2, \ldots\ldots, x_n\}$. ∴ $X-\{x_1, x_2, \ldots\ldots, x_n\}$ is τ^{*b}open(by theorem (8.13).∴$U \cap [X-\{x_1, x_2, \ldots\ldots, x_n\}]$ is τ^{*b}open neighborhood of x and it does not intersect A which is a contradiction.

Theorem 8.18: Every finite subset of a finitely *b-additive bT_1 modulo $\mathcal{I}$ space has no b-limit point modulo $\mathcal{I}$.

Proof: The proof follows from theorem (8.17).

Theorem 8.19: If $(X, \tau, \mathcal{I})$ is a finitely *b-additive,T_1 space then (X, τ^{*b}) is a T_1 space.

Proof: The proof follows from the theorem (8.9), since every T_1 space is bT_1 modulo $\mathcal{I}$. But, the converse is not true.

Example 8.20: Let $(X, \tau, \mathcal{I})$ be an indiscrete ideal topological space where $\mathcal{I} = \mathcal{P}(X)$.

Then, $cl^{*b}(A) = A \ \forall\ A \subseteq X$. $\therefore \tau^{*b}$ is discrete topology and hence (X, τ^{*b}) is a *b-additive T_1space .But, (X, τ) is not a T_1 space.

Theorem 8.21: Let f: $(X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, f(\mathcal{I}))$ be a bijection.

(1) f is b-open and X is T_1 modulo $\mathcal{I} \Rightarrow$ Y is bT_1 modulo $f(\mathcal{I})$
(2) f is b-resolute and X is bT_1 modulo $\mathcal{I} \Rightarrow$ Y is bT_1 modulo $f(\mathcal{I})$
(3) f is b-continuous and Y is T_1 modulo $f(\mathcal{I}) \Rightarrow$ X is bT_1 modulo $\mathcal{I}$.
(4) f is b-irresolute and Y is bT_1 modulo $f(\mathcal{I}) \Rightarrow$ X is bT_1 modulo $\mathcal{I}$.
(5) f is strongly b-continuous and Y is bT_1 modulo $f(\mathcal{I}) \Rightarrow$ X is T_1 modulo $\mathcal{I}$.

Proof: Similar to the proof of theorem (7.11).

Remark 8.22: The property of being a ' bT_1 space modulo $\mathcal{I}$ 'is a b-topological property by (2) and (4) of theorem (8.21).

9.bT_2Modulo an ideal or b-Hausdorff space

Definition 9.1: An ideal topological space $(X, \tau, \mathcal{I})$ is said to be bT_2 modulo $\mathcal{I}$ if for every $x \neq y \in X$, there exist b-open sets U,V in X such that $x \in U\backslash V$, $y \in V\backslash U$ and $U \cap V \in \mathcal{I}$.

Example 9.2: Let X be infinite set and τ-cofinite topology, $\mathcal{I} = \mathcal{P}(X)$.

Then BO(X)= { φ, X ,A/ A is infinite}. Let $x \neq y \in X$ and let U =X-{x}; V=X-{y};U,V are b-open in X, y∈ U, x ∈V and U∩V= {X-{x} ∩Y-{y}}=X-{x,y} ∈$\mathcal{I}$.∴ X is bT_2 modulo $\mathcal{I}$.

Example 9.3: In example (8.3), (X,τ, φ) is not bT_2 modulo φ .

Remark 9.4:

(1) Every T_2 space is T_2 modulo $\mathcal{I}$ for any ideal $\mathcal{I}$.

(2) A space is bT_2 space ⇔ it is bT_2 modulo {φ}

(3) (X, τ, $\mathcal{I}$) is bT_2 modulo $\mathcal{I}$ and $\mathcal{I} \subseteq J$ ⇒ (X, τ, J) is bT_2 modulo J.

(4) (X, τ, $\mathcal{I}$) is T_2 modulo $\mathcal{I}$ ⇒ (X, τ, $\mathcal{I}$) is bT_2 modulo $\mathcal{I}$.

Theorem 9.5: Let (X, τ, $\mathcal{I}$) be finitely *b-additive.Then (X, τ, $\mathcal{I}$) is bT_2 modulo $\mathcal{I}$ ⇒ (X, τ^{*b}) is a T_2 space. The converse is true if (X, τ, $\mathcal{I}$) is finitely b-additive.

Proof: Similar to the proof of theorem (8.9).

Theorem 9.10: Every bT_2 modulo $\mathcal{I}$ space is bT_1 modulo $\mathcal{I}$ space.

Proof: Obvious from the definition

Example 9.11: An indiscrete space(X, τ)is not T_2 but it is bT_2 and bT_2 modulo $\mathcal{I}$ for any ideal $\mathcal{I}$

Theorem 9.12: Every finite subset of a finitely *b-additive, bT_2 modulo $\mathcal{I}$ space (X, τ, $\mathcal{I}$) is τ^{*b} closed.

Proof: By the theorem (9.10), (X, τ, $\mathcal{I}$) is bT_1 modulo $\mathcal{I}$ and by the theorem (8.9), (X, τ^{*b}) is a T_1 space and hence every singleton set is τ^{*b} –closed, since X is finitely *b-additive, every finite subset is τ^{*b} –closed.

Definition 9.13:

1. A τ^{*b} –open set U containing x is called a τ^{*b} neighborhood of x.

2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\{x_n\}$ a sequence in X. We say $x_n \overset{b}{\rightarrow} x$ (modulo $\mathcal{I}$) if for every τ^{*b} neighborhood U of x there exist positive integer N such that $x \in U$ for all $n \geq N$

In this case we say x is b- limit modulo $\mathcal{I}$ of the sequence $\{x_n\}$.

Theorem 9.14: Let $(X, \tau, \mathcal{I})$ be a finitely *b-additive T_2 modulo $\mathcal{I}$ space and $\{x_n\}$ a sequence in X. If b- limit modulo $\mathcal{I}$ of the sequence $\{x_n\}$ exists then it is unique.

Proof: Suppose that $x_n \overset{b}{\rightarrow} x$ modulo $\mathcal{I}$ and $x_n \overset{b}{\rightarrow} y$ modulo $\mathcal{I}$ and $x \neq y$, since X is bT_2 modulo $\mathcal{I}$ there exist U, V in BO (X) such that $x \in U \setminus V$, $y \in V \setminus U$, $U \cap V \in \mathcal{I}$

Let $I = U \cap V$, then $V - I \in \tau^{*b}$ (since V-I is a sub basic open set in τ^{*b} topology) and $U \cap (V - I) = \varphi$. ∴U contains all the elements of $\{x_n\}$ except finite number of points.Hence,

$V - I$ contain only finite number of points of $\{x_n\}$.Therefore, it is a contradiction to the fact $x_n \overset{b}{\rightarrow} y$ modulo $\mathcal{I}$.

Theorem 9.15: If $(X, \tau, \mathcal{I})$ is a finitely *b additive, T_2 space, then (X, τ^{*b}) is a T_2 space.

Proof: Since $(X, \tau, \mathcal{I})$ is finitely *b-additive, τ^{*b} is a topology. For $x \neq y \in X$, there exist

U,V in τ such that $x \in U$, $y \in V$ and $U \cap V = \varphi$. Since $\tau \subseteq \tau^{*b}$, $U, V \in \tau^{*b}$. ∴(X, τ^{*b}) is a T_2 space.

Remark 9.16: Converse of the above theorem need not be true as seen from the following example

Example 9.17 : Consider $(X, \tau, \mathcal{I})$ where τ is an indiscrete topology and $\mathcal{I} = \mathcal{P}(X)$

Then $A^{*b} = \varphi$ for all $A \subseteq X$. $cl^{*b}(A) = A \cup A^{*b} = A$. Therefore, every subset is closed in τ^{*b}

topology. τ^{*b} is discrete topology. ∴(X, τ^{*b}) is a finitely *b-additive, T_2 space.But (X, τ) is not a T_2 space.

Theorem 9.18: Let (X, τ, $\mathcal{I}$) be a finitely *b additive, bT_2 modulo $\mathcal{I}$ space then ,

(1) For each pair x, y ∈X there exist closed τ^{*b} neighbourhood N_y of y such that x∉ N_y(N_y is said to be a closed τ^{*b} neighbourhood , if N_y is τ^{*b} –closed and there exist a τ^{*b} –open set V such that V⊆N_y)

(2) For x∈X, {x}=∩ N_x where the intersection is over τ^{*b}closed neighbourhood $N_x$of x.

Proof:

(1) X is bT_2modulo $\mathcal{I}$ space ⇒(X, τ^{*b}) is a T_2 space (by theorem (9.5))

Let x≠ y ∈ X. Then there exist U,V in τ^{*b}such that x∈U,y∈V and U∩V=φ

Then, N_y= X\Uis a τ^{*b}–closed set such that y∈V ⊆$N_y$and x∉ N_y

(2) Let x ∈X ,y≠ x. Then by (1),there exist a τ^{*b}closed neighborhood N_x of x such that y∉N_x

Therefore, y ∉ intersection of all closedτ^{*b}neighborhood of x.Therefore, {x} =∩ N_x.

Theorem 9.19: Let f: (X, τ, $\mathcal{I}$) →(Y, σ, f($\mathcal{I}$)) be a bijection.

(1) f is b-open and X is T_2 modulo $\mathcal{I}$⇒Y is bT_2 modulo f($\mathcal{I}$)
(2) f is b-resolute and X is bT_2 modulo $\mathcal{I}$⇒Y is bT_2 modulo f($\mathcal{I}$)
(3) f is b-continuous and Y is T_2 modulo f($\mathcal{I}$) ⇒X is bT_2 modulo $\mathcal{I}$.
(4) f is b- irresolute and Y is bT_2 modulo f($\mathcal{I}$) ⇒X is bT_2 modulo $\mathcal{I}$.
(5) f is strongly b-continuous and Y is bT_2 modulo f($\mathcal{I}$) ⇒X is T_2 modulo $\mathcal{I}$.

Proof: Similar to the proof of theorem (7.11).

Remark 9.20: The property of being "bT_2modulo $\mathcal{I}$" is a b-topological property by (2) and (4) of theorem (9.20).

Theorem 9.21: Let (X, τ, $\mathcal{I}$) be finitely b-additive, bT_2 modulo $\mathcal{I}$, x ∈X and C a b-compact subspace of X not containing x, then there exist U,V such that V ∈ BO(X), U is a finite intersection of b-open sets such that, x ∈U, C ⊂ V and U∩V ∈$\mathcal{I}$.

Proof: For every y∈ C, there exists G_y , H_y∈ BO(X) such that x ∈$G_y \setminus H_y$, y ∈ $H_y \setminus G_y$,

$G_y \cap H_y \in \mathcal{I}$.∴ $\{H_y / y \in C\}$ is a b-open cover for C.Since C is b-compact there exist

H_{y_i},…………..,H_{y_n} such that C $\subseteq \bigcup_{i=1}^{n} H_{y_i}$.Let G_{y_1},……., G_{y_n} be the corresponding b-open sets containing x. Let U=$\cap_{i=1}^{n} G_{y_i}$; V= $\bigcup_{i=1}^{n} H_{y_i}$.Finite union of b-open sets is b-open.∴ V is b-open. Now, x ∈ V\U, C⊆U\V; Put G_{y_i} =G_i and $H_{y_i} = H_i$.Then, U∩V

$=(\bigcap_{i=1}^{n} G_i) \cap (\bigcup_{i=1}^{n} H_i)= (G_1 \cap G_2 \cap \ldots\ldots\ldots \cap G_n) \cap (H_1 \cup H_2 \cup \ldots\ldots. \cup H_n)$

$= \{(G_1 \cap G_2 \cap \ldots\ldots\ldots \cap G_n) \cap H_1\} \cup \{(G_1 \cap G_2 \cap \ldots\ldots\ldots \cap G_n) \cap H_2\} \cup \ldots\ldots\ldots\ldots.. \cup$

$(G_1 \cap G_2 \cap \ldots\ldots\ldots \cap G_n) \cap H_n\} \subseteq (G_1 \cap H_1) \cup (G_2 \cap H_2) \cup \ldots\ldots.. \cup (G_n \cap H_n)= \bigcup_{i=1}^{n}(G_i \cap H_i)$

$\in \mathcal{I}$. ∴ U∩V $\in \mathcal{I}$.

Remark 9.22: In the above theorem, suppose (X, τ, $\mathcal{I}$) is finitely b-additive then U in the above proof is b-open.

Theorem 9.23: Let (X, τ, $\mathcal{I}$) be a finitely *b-additive,bT_2 modulo $\mathcal{I}$ space. Then every b-compact subspace of X is τ^{*b} closed.

Proof: Let C be a b-compact subspace of a bT_2 modulo $\mathcal{I}$ finitely *b-additive space X.

Claim: C is τ^{*b} closed.(i.e) To prove: C^c is τ^{*b} open.Let x∈ C^c,by theorem (9.21), there exist U,V such that V∈BO(X) and U is finite intersection of b-open sets, x ∈ U\V, C⊆V\U and U∩V ∈$\mathcal{J}$. Let U∩V= I. Now, U-I is a basic b-open set and hence U-I∈τ^{*b}.

Now, U=$\bigcap_{i=1}^{n} G_i$ where G_i∈ BO(X)

Now, x ∈ U\I ⊆ X\V ⊆ X\C.∴ X-C is τ^{*b} open and hence C is τ^{*b} closed.

Remark 9.24: The following example shows that, the converse of the above theorem is not true

Example 9.25: Let (X, τ) be an infinite discrete space and $\mathcal{J}$= $\mathcal{P}$(X). Then BO(X)={all subsets} and $\tau^{*b} = \{\text{all subsets}\}$ .This space is finitely *b-additive, bT_2 modulo $\mathcal{J}$. An infinite subset B of X is τ^{*b} -closed but not b-compact since {{x}/x∈ X} is a b-open cover for B which has no finite sub cover.

10.bT_3 modulo an idealor b-regular modulo an ideal

Definition 10.1: An ideal topological space (X, τ, $\mathcal{J}$) is said to be a bT_3modulo $\mathcal{J}$ space or b-regular modulo an ideal space, if

(1)X is bT_1modulo $\mathcal{J}$ and

(2)For any b-closed subset F of X and every point x ∉F there exist G, H ∈BO(X) such that x ∈H\G, F⊆G-H and G∩H ∈$\mathcal{J}$.

Note 10.2: (X, τ) is b-regular ⇔ (X, τ, $\mathcal{J}$) is b-regular modulo {φ}.

Theorem 10.3: Let (X, τ, $\mathcal{J}$) be a finitely *b additive. Then (X, τ, $\mathcal{J}$) is b-regular modulo $\mathcal{J}$ ⇒(X, τ^{*b}) is regular. The converse need not be true.

Proof:(X, τ, $\mathcal{J}$) is bT_1modulo $\mathcal{J}$⇒(X, τ^{*b}) is a T_1 space. Let F be b-closed set in X and x ∉ F, then by definition there exist G,H∈BO(X) such that F ⊆ G\H and x ∈ H\G and G∩H∈$\mathcal{J}$. Let G∩H = I, Put G'=G-I and H'=H-I, then $G', H'$are sub basic$\tau^{*b} -$ open sets and F⊆G',x∈$H'$and $G' \cap H'$ =φ .Therefore, (X, τ^{*b}) is regular.

Example 10.4: Let (X, τ, $\mathcal{J}$) be an indiscrete ideal topological space where $\mathcal{J}$=$\mathcal{P}$(X). Then τ^{*b} is discrete topology .Then BO(X) = {All subsets}.τ^{*b} = {All subsets}. (X, τ, $\mathcal{J}$) is finitely

*b additive .So, (X, τ^{*b}) is regular .

Example 10.5: Let (X, τ) be a discrete ideal topological space thenBO(X) =$\mathcal{P}$(X). Then

(X, τ) is b-regular. So, (X, τ, $\mathcal{J}$) is b-regular modulo $\mathcal{J}$ for any ideal $\mathcal{J}$.

Theorem 10.6: Every b- regular modulo $\mathcal{J}$ space is bT_2modulo $\mathcal{J}$ for any ideal $\mathcal{J}$.

Proof: Let (X, τ, $\mathcal{J}$) be b-regular modulo $\mathcal{J}$. To Prove: (X, τ, $\mathcal{J}$) is a bT_2modulo $\mathcal{J}$. Let x ≠y ∈ X, (X, τ, $\mathcal{J}$) is b-regular modulo $\mathcal{J}$⇒ (X, τ, $\mathcal{J}$) is bT_1modulo $\mathcal{J}$. Then there exists b-open sets U and V such that x ∈U\V and y∈ V\U, U∩V ∈$\mathcal{J}$.X-V is a b-closed set and y∉ X\V.

Since (X, τ, $\mathcal{J}$) is b-regular modulo $\mathcal{J}$, there exists b-open sets W_1,W_2 such that y ∈W_1\W_2, x ∈ X-V ⊆W_2 -W_1 and W_1∩W_2∈$\mathcal{J}$. Hence (X, τ, $\mathcal{J}$) is bT_2modulo $\mathcal{J}$.

Theorem 10.7: Let (X, τ, $\mathcal{J}$) be finitely *b additive. If X is b-regular modulo $\mathcal{J}$ then given a point x∈ X and a b-open set U containing x, there exist b-open set V containing x such that x ∈V ⊆cl^{*b}(V) ⊆ U.

Proof: Let X be finitely semi *b additive space. Then τ^{*b} is a topology.Let X be b-regular modulo $\mathcal{J}$; Let U ∈ BO(X) and x ∈ U . Then F=X-U is b-closed and x ∉ F. By definition, there exist G, H ∈ BO(X) such that F⊆ G-H, x ∈H\G, G∩H∈$\mathcal{J}$. Let G∩H=I. Put G' =G-I, H'=H-I.Then G'andH' are sub basic open sets in τ^{*b}–topology.∴ $G'$and $H' \in \tau^{*b}$ $and G' \cap H' = \varphi$ Now, x ∈H'⊆$cl^{*b}(H')$.To prove: $cl^{*b}(H') \subseteq$ U. If y∈ F then y∈G' which is disjoint fromH'.∴G' is a τ^{*b} neighborhood of y, disjoint from H'.∴ y $\notin cl^{*b}(H')$. ∴F⊆ X- $cl^{*b}(H')$. ∴U ⊇$cl^{*b}(H')$.∴ x ∈H'⊆$cl^{*b}(H') \subseteq$ U. Hence the proof.

Remark 10.8: Let f:(X, τ, $\mathcal{J}$)→(Y, σ, f($\mathcal{J}$)) be a bijection.

(1) f is b-open and X is regular modulo $\mathcal{J}$⇒Y is b-regular modulo f($\mathcal{J}$)

(2) f is b-resolute and X is b-regular modulo $\mathcal{J}$⇒Y is b-regular modulo f($\mathcal{J}$)

(3) f is b-continuous and Y is regular modulo f($\mathcal{J}$) ⇒X is b-regular modulo $\mathcal{J}$.

(4) f is b-irresolute and Y is b-regular modulo f($\mathcal{J}$) ⇒X is b-regular modulo $\mathcal{J}$.

(5)f is strongly b-irresolute and Y is b-regular modulo f($\mathcal{J}$) ⇒X is b-modulo $\mathcal{J}$.

Proof: Similar to the proof of theorem (7.12).

Remark 10.9: From(2) and (3) of theorem (10.8) it follows that 'b-regular modulo $\mathcal{J}$' is a

b-topological property.

11. b-NORMAL MODULO AN IDEAL

Definition 11.1: An ideal topological space (X, τ, $\mathcal{J}$) is said to be b-normal modulo $\mathcal{J}$ if

(1) X is bT_1modulo$\mathcal{J}$.

(2) For any two disjoint b-closed sets G, H in X, there exist U, V ∈BO(X) such

that G ⊆U\V, H ⊆V\U and G∩H ∈$\mathcal{I}$.

Example 11.2: Let (X, τ, $\mathcal{I}$) be a discrete ideal topological space where $\mathcal{I}$= {φ}. Then

BO(X) = {all subsets}.∴This space is b-normal modulo $\mathcal{I}$.

Example 11.3: In example (8.3), (X,τ, $\mathcal{I}$) is not bT_1 modulo $\mathcal{I}$ and hence it is not b-normal

Modulo $\mathcal{I}$.

Remark 11.4:

(1) (X, τ, $\mathcal{I}$) is b-normal modulo $\mathcal{I}$ and $\mathcal{I}$⊆ J ⇒(X, τ, J) is b-normal modulo J.

(2) (X, τ) is b- normal ⇒(X, τ, $\mathcal{I}$) is b-normal modulo $\mathcal{I}$ for any ideal.

(3) (X, τ) is b-normal ⇔(X, τ, φ) is b-normal modulo {φ}

Theorem 11.5: Let (X, τ, $\mathcal{I}$) be finitely *b additive, b-normal modulo $\mathcal{I}$ space. Then (X,τ^{*b}) is normal.

Proof: By definition, (X, τ, $\mathcal{I}$) is a bT_1modulo$\mathcal{I}$ space. Under the given hypothesis, τ^{*b} is a topology and $\mathcal{B}$ = {V-I/V∈BO(X), I∈$\mathcal{I}$} is a sub basis for the topology. By the theorem (3.6), every singleton set is τ^{*b}–closed.Let G,H be two disjoint b-closed sets.Then by the hypothesis there exist U,V∈BO(X) such that G ⊆ U\V, H ⊆ V\U andU∩V ∈$\mathcal{I}$.Let U∩V = I, Put G'=U-I and H'=V-I,then $G', H'$are sub basic open sets in τ^{*b}-topology.∴G' and H'∈τ^{*b}. Now, G ⊆G', H ⊆H' and $G' \cap H'$ = φ.∴ (X,τ^{*b}) is normal.

Remark 11.6: Let f:(X, τ, $\mathcal{I}$)→(Y, σ, f($\mathcal{I}$)) be a bijection.

(1) f is b-open and X is normal modulo $\mathcal{I}$⇒Y is b-normal modulo f($\mathcal{I}$)

(2) f is b-resolute and X is b-normal modulo $\mathcal{I}$⇒Y is b-normal modulo f($\mathcal{I}$)

(3) f is b-continuous and Y is normal modulo f($\mathcal{I}$) ⇒X is b-normal modulo $\mathcal{I}$.

(4) f is b-irresolute and Y is b-normal modulo f($\mathcal{I}$) ⇒X is b-normal modulo $\mathcal{I}$.

(5) f is strongly b-irresolute and Y is b-normal modulo f($\mathcal{I}$) ⇒X is normal modulo $\mathcal{I}$.

Proof: Similar to the proof of theorem (7.11)

Note 11.7: The property of being 'b-normal modulo $\mathcal{I}$' is b-topological property.

Theorem 11.8: Let (X, τ, $\mathcal{I}$) be finitely *b additive, finitely b-additive, b-compact space which is bT_2 modulo $\mathcal{I}$. Then (X, τ, $\mathcal{I}$) is b-normal modulo $\mathcal{I}$.

Proof: Let (X, τ) be b-compact and bT_2 modulo $\mathcal{I}$.Then, X is bT_1 modulo $\mathcal{I}$. Let G, H be disjoint b-closed sets in X.First let us prove G and H are b-compact. Let $\{A_\alpha\}$ be a b-open cover for G. Then,{{ A_α}/ X-G} is a b-open cover for X.Since X is b-compact, there existα_1, α_2,……..,α_n such that X= $A_{\alpha_1}\cup$…………$\cup A_{\alpha_n}\cup$ X-G.

Then G= $A_{\alpha_1}\cup$…………$\cup A_{\alpha_n}$.∴ G is b-compact.Similarly H is b-compact.For each x∈ G there exist U_x and V_x∈BO(X) such that x ∈U_x ; H ⊆$V_x$and $U_x\cap V_x\in\mathcal{I}$(by theorem(9.21).{$U_x$/x∈G} is a b-open cover for G. Since G is b-compact there existx_1,……..,x_n such that G⊆$\bigcup_{i=1}^n U_{x_i}$. Let U = $\bigcup_{i=1}^n U_{x_i}$ and V = $\bigcap_{i=1}^n V_{x_i}$U and V are b-open(Since X is finitely b-additive). Then, G ⊆ U, H⊆Vand U ∩ V = ($\bigcup_{i=1}^n U_{x_i}$) ∩ ($\bigcap_{i=1}^n V_{x_i}$) = ($V_{x_1}\cap$…..$\cap V_{x_n}$)$\cap U_{x_1}$}∪….∪{($V_{x_1}\cap$……$\cap V_{x_n}$)$\cap U_{x_n}$}⊆($V_{x_1}\cup U_{x_1}$)∪…….∪($V_{x_n}\cap U_{x_n}$) ∈$\mathcal{I}$.∴U ∩ V ∈$\mathcal{I}$.∴ X is b-normal modulo $\mathcal{I}$.

Note 11.9: But converse need not be true as seen from the following example.

Example 11.10: Let (X, τ, $\mathcal{I}$) be an infinite discrete ideal topological space where

$\mathcal{I}$= $\mathcal{P}$(X). Then BO(X) = {all subsets},τ^{*b} = {all subsets}.∴ (X, τ, $\mathcal{I}$) is finitely *b additive, finitely b-additive. (X, τ, $\mathcal{I}$) is a bT_2modulo $\mathcal{I}$. But it is not b-compact {{x}/x ∈ X} is a b-open cover for X which has no finite subcover.

Theorem 11.11: Let (X, τ, $\mathcal{I}$) is finitely *b additive space. Then if X is b-normal modulo $\mathcal{I}$ then given a b-closed set A and a b-open set U containing A, there exist a b-open set G containing A, such that A⊆ G⊆cl^{*b}G ⊆ U.

Proof: Under the given hypothesis, τ^{*b} is a topology and $\mathcal{B}$ = {V-I/V∈BO(X), I∈$\mathcal{I}$} is a sub basis for τ^{*b}. A and X-U are two disjoint τ^{*b}–closed sets.Since X is b-normal modulo $\mathcal{I}$ there exist V and W ∈BO(X) such that A ⊆ V\W, X\U ⊆ W\V and V ∩ W = I ∈$\mathcal{I}$. Put G = V-I and H = W-I. Then G and H are sub basic open sets and hence G and H are inτ^{*b}.Now, A⊆ G and X-U ⊆H and G∩H = φ. ∴G⊆X-H ⊆U .X-H is τ^{*b}–closed.

∴cl^{*b}(X-H)= X-H. A ⊆ G ⊆cl^{*b} (G) ⊆ X-H ⊆U.Hence the proof.

Theorem 11.12: Let (X, τ, $\mathcal{I}$) be finitely *b additive space with the following conditions.

(1) τ^{*b} = BO(X).

(2) (X, τ, $\mathcal{I}$) is a bT_1modulo $\mathcal{I}$.

(3) Given a b- closed set A and a b-open set U containing A, there exist a b-open set G containing A, such that A⊆ G⊆cl^{*s}G ⊆ U.

Then (X, τ, $\mathcal{I}$) is b-normal modulo $\mathcal{I}$.

Proof: Let A and B be two disjoint b-closed sets in X.Then U= X-B is a b-open set containing A. By the hypothesis, there exist a b-open set G such that A⊆ G⊆cl^{*b}G ⊆ Usinceτ^{*b} = BO(X),G and H= X- cl^{*b}G are b-open. Now, A⊆ G, B⊆ H and G∩H =φ ∈$\mathcal{I}$. ∴(X, τ, $\mathcal{I}$) is b- normal modulo $\mathcal{I}$.

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