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# A New Approach to Solution of Fuzzy Differential Equations

Dr.C. Duraisamy, B. Usha\*

Department of Mathematics, Kongu Engineering College,  
Perundurai, Erode-638 052, Tamilnadu, India.

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## Abstract

In this paper, numerical algorithms for solving “fuzzy ordinary differential equations” are considered. A scheme based on the Runge-Kutta method of order two is discussed in detail and this is followed by a complete error analysis.

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**Keywords:** Fuzzy differential equations, Runge-Kutta method of order two, Trapezoidal fuzzy number.

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## 1 Introduction

The study of fuzzy differential equations (FDEs) forms a suitable setting for model dynamical systems in which uncertainties or vagueness pervade. First order linear and non-linear FDEs are one of the simplest FDEs which appear in many applications. In the recent years, the topic of FDEs has been investigated extensively. The organized of the paper is as follows. In the first three sections below, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. In section five, we present Runge-Kutta method of order two and its iterative solution for solving of Fuzzy differential equations. The proposed algorithm is illustrated by an example in the last section.

## 2 Preliminary

A trapezoidal fuzzy number  $u$  is defined by four real numbers  $k < l < m < n$ ,

where the base of the trapezoidal is the interval  $[k, n]$  and its vertices at  $x = l, x = m$ . Trapezoidal fuzzy number will be written as  $u = (k, \ell, m, n)$ . The membership function for the trapezoidal fuzzy number  $u = (k, \ell, m, n)$  is defined as the following :

$$u(x) = \begin{cases} \frac{x - k}{l - k}, & k \leq x \leq l \\ 1, & l \leq x \leq m \\ \frac{x - n}{m - n}, & m \leq x \leq n \end{cases} \quad (1)$$

we will have :  $u > 0$  if  $k > 0$ ;  $u > 0$  if  $l > 0$ ;  $u > 0$  if  $m > 0$ ; &  $u > 0$  if  $n > 0$ .

Let us denote  $R_F$  by the class of all fuzzy subsets of  $R$  (i.e.  $u : R \rightarrow [0,1]$ ) satisfying the following properties:

- (i)  $\forall \square u \in R_F$ ,  $u$  is normal, i.e.  $\exists \square x_0 \in R$  with  $u(x_0) = 1$ ;
- (ii)  $\forall \square u \in R_F$ ,  $u$  is convex fuzzy set (i.e.  $u(tx + (1-t)y) \geq \min \{u(x), u(y)\}$ ,  
 $\forall \square t \in [0,1], x, y \in R$ );
- (iii)  $\forall \square u \in R_F$ ,  $u$  is upper semi continuous on  $R$ ;
- (iv)  $\{\overline{x \in R | u(x) > c}\}$  is compact, where  $\overline{A}$  denotes the closure of  $A$ .

Then  $R_F$  is called the space of fuzzy numbers. Obviously  $R \subset R_F$ . Here  $R \subset R_F$  is understood as  $R = \{x \in R | u(x) = 1\}$ . We define the  $r$ -level set,  $x \in R$ ;

$$[u]_r = \{x \in R | u(x) \geq r\}, \quad 0 \leq r \leq 1; \quad (2)$$

Clearly,  $[u]_r = \{x \in R | u(x) \geq r\}$  is compact,

which is a closed bounded interval and we denote by  $[u]_r = [\underline{u}(r), \overline{u}(r)]$ . It is clear that the following statements are true,

1.  $\underline{u}(r)$  is a bounded left continuous non decreasing function over  $[0,1]$ ,
2.  $\overline{u}(r)$  is a bounded right continuous non increasing function over  $[0,1]$ ,
3.  $\underline{u}(r) \leq \overline{u}(r)$  for all  $r \in (0,1]$ ,

for more details see [2],[3].

$$\text{Let } D: R_F \times R_F \rightarrow R_+ \cup \{0\},$$

$D(u, v) = \sup_{r \in [0,1]} \max \{ |\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)| \}$  be Hausdorff distance between fuzzy numbers, where  $[u]_r = [\underline{u}(r), \bar{u}(r)]$ ,  $[v]_r = [\underline{v}(r), \bar{v}(r)]$ . The following properties are well-known:

$$D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in R_F,$$

$$D(k.u, k.v) = |k|D(u, v), \quad \forall k \in R, u, v \in R_F,$$

$$D(u + v, w + e) \leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in R_F$$

and  $(R_F, D)$  is a complete metric space.

**Lemma 2.1** If the sequence of non-negative numbers

$\{W_n\}_{n=0}^N$  satisfy

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N-1,$$

for the given positive constants  $A$  and  $B$ , then

$$|W_n| \leq A^n |W_0| + B \frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.$$

**Lemma 2.2** If the sequence of numbers  $\{W_n\}_{n=0}^N$ ,  $\{V_n\}_{n=0}^N$  satisfy

$$|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B,$$

for the given positive constants  $A$  and  $B$ , then denoting

$$U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N,$$

we have,  $U_n \leq \bar{A}^n U_0 + \bar{B} \frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \leq n \leq N,$

where  $\bar{A} = 1 + 2A$  and  $\bar{B} = 2B$ .

### 3 Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \quad (3)$$

where  $y$  is a fuzzy function of  $t$ ,  $f(t, y)$  is a fuzzy function of the crisp variable  $t$  and the fuzzy variable  $y$ ,  $y'$  is the fuzzy derivative of  $y$  and  $y(t_0) = y_0$  is a trapezoidal or a trapezoidal shaped fuzzy number.

We denote the fuzzy function  $y$  by  $y = [\underline{y}, \bar{y}]$ . It means that the  $r$ -level set of  $y(t)$  for  $t \in [t_0, T]$  is

$$[y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)], \quad [y(t_0)]_r = [\underline{y}(t_0; r), \bar{y}(t_0; r)], \quad r \in (0, 1]$$

we write  $f(t, y) = [\underline{f}(t, y), \bar{f}(t, y)]$  and

$$\underline{f}(t, y) = F[t, \underline{y}, \bar{y}], \quad \bar{f}(t, y) = G[t, \underline{y}, \bar{y}].$$

Because of  $y' = f(t, y)$  we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \bar{y}(t; r)] \quad (4)$$

$$\bar{f}(t, y(t); r) = G[t, \underline{y}(t; r), \bar{y}(t; r)] \quad (5)$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\}, \quad s \in R \quad (6)$$

so fuzzy number  $f(t, y(t))$ . From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \bar{f}(t, y(t); r)], \quad r \in (0, 1], \quad (7)$$

where 
$$\underline{f}(t, y(t); r) = \min \{f(t, u) | u \in [y(t)]_r\} \quad (8)$$

$$\bar{f}(t, y(t); r) = \max \{f(t, u) | u \in [y(t)]_r\}. \quad (9)$$

**Definition 3.1** A function  $f: R \rightarrow R_F$  is said to be fuzzy continuous function, if for an arbitrary fixed  $t_0 \in R$  and  $\epsilon > 0$ ,  $\delta > 0$  such that

$$|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \epsilon \quad \text{exists.}$$

Throughout this paper we also consider fuzzy functions which are continuous in metric  $D$ . Then the continuity of  $f(t, y(t); r)$  guarantees the existence of the definition of  $f(t, y(t); r)$  for  $t \in [t_0, T]$  and  $r \in [0, 1]$  [8]. Therefore, the functions  $G$  and  $F$  can be definite too.

#### 4 Runge- Kutta method of order two

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \quad (10)$$

It is known that, the sufficient conditions for the existence of a unique solution to (10) are that  $f$  to be continuous function satisfying the Lipschitz condition of the following form:

$$\| f(t, x) - f(t, y) \| \leq L \| x - y \|, \quad L > 0.$$

We replace the interval  $[t_0, T]$  by a set of discrete equally spaced grid points,

$$t_0 < t_1 < t_2 < \dots < t_N = T, \quad h = \frac{T - t_0}{N}, \quad t_i = t_0 + ih, \quad i = 0, 1, \dots, N$$

to obtain the Euler's method for the system (10), we apply Trapezoidal numerical integration method.

By the mean value theorem of integral calculus, we obtain

$$y(t_{n+1}) = y(t_n) + hf(t_n + \theta h, y(t_n + \theta h)) + O(h^3), \quad 0 < \theta < 1 \quad (11)$$

If we approximate  $y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n))$  in the argument of  $f$ , we get

$$y(t_{n+1}) = y(t_n) + hf(t_{n+1}, y(t_n)) + hf(t_n, y(t_n)) + O(h^3) \quad (12)$$

If we set

$$\begin{aligned} K_1 &= hf(t_n, y(t_n)) \\ K_2 &= hf(t_{n+1}, y(t_n) + K_1) \end{aligned}$$

$$\text{We get the method as } y(t_{n+1}) = y(t_n) + K_2 + O(h^3) \quad (13)$$

when  $\theta = 1/2$ , we obtain

$$y(t_{n+1}) = y(t_n) + hf(t_n + h/2, y(t_n + h/2)) + O(h^3) \quad (14)$$

However,  $t_n + (h/2)$  is not a nodal point.

If we approximate  $y(t_n + h/2)$  in (14) by Euler's method with spacing  $h/2$ , we get,

$$y(t_n + h/2) = y(t_n) + h/2 f(t_n, y(t_n)) + O(h^3)$$

Then, we have the approximation

$$y(t_{n+1}) = y(t_n) + hf(t_n + h/2, y(t_n) + h/2 f(t_n, y(t_n))) + O(h^3) \quad (15)$$

If we set

$$\begin{aligned} K_1 &= hf(t_n, y(t_n)) \\ K_2 &= hf(t_n + h/2, y(t_n) + K_1/2) \end{aligned}$$

$$\text{then (15) can be written as, } y(t_{n+1}) = y(t_n) + K_2 \quad \text{as } h \rightarrow 0. \quad (16)$$

Alternately, if we use the approximation

$$y'(t_n + h/2) = 1/2 (y'(t_n) + y'(t_n + h))$$

and the Euler method, we obtain

$$y'(t_n + h/2) = 1/2 (f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n, y(t_n))))$$

Thus (14) may be approximated by

$$y(t_{n+1}) = y(t_n) + h/2 (f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n, y(t_n)))) \quad (17)$$

If we set

$$\begin{aligned} K_1 &= hf(t_n, y(t_n)) \\ K_2 &= hf(t_{n+1}, y(t_n) + K_1) \end{aligned}$$

then (17) can be written as,

$$y(t_{n+1}) = y(t_n) + 1/2 (K_1 + K_2) \quad (18)$$

The proposed algorithm is illustrated by an example in last section.

## 5 Runge- Kutta method of order two for solving Fuzzy Differential Equations

Let  $Y = [\underline{Y}, \bar{Y}]$  be the exact solution and  $y = [\underline{y}, \bar{y}]$  be the approximated solution of the fuzzy initial value problem (3).

$$\text{Let } [Y(t)]_r = [\underline{Y}(t; r), \bar{Y}(t; r)], \quad [y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)].$$

Throughout this argument, the value of  $r$  is fixed. Then the exact and approximated solution at  $t_n$  are respectively denoted by

$$[Y(t_n)]_r = [\underline{Y}(t_n; r), \bar{Y}(t_n; r)], \quad [y(t_n)]_r = [\underline{y}(t_n; r), \bar{y}(t_n; r)] \quad (0 \leq n \leq N).$$

The grid points at which the solution is calculated are

$$\mathbf{r} = \frac{\mathbf{F}\mathbf{t}_0}{N}, \mathbf{t}_i = \mathbf{t}_0 + \mathbf{i}, \quad 0 \leq i \leq N$$

Then we obtain ,  $\underline{Y}(t_{n+1}; r) = \underline{Y}(t_n; r) + \frac{1}{2}[K_1 + K_2]$

where

$$\begin{aligned} K_1 &= hF[t_n, \underline{Y}(t_n; r), \bar{Y}(t_n; r)] \\ K_2 &= hF[t_{n+1}, \underline{Y}(t_n; r), \bar{Y}(t_n; r) + K_1] \end{aligned} \quad (19)$$

and

$$\bar{Y}(t_{n+1}; r) = \bar{Y}(t_n; r) + \frac{1}{2}[K_1 + K_2]$$

where

$$\begin{aligned} K_1 &= hG[t_n, \underline{Y}(t_n; r), \bar{Y}(t_n; r)] \\ K_2 &= hG[t_{n+1}, \underline{Y}(t_n; r), \bar{Y}(t_n; r) + K_1] \end{aligned} \quad (20)$$

And we have

$$\underline{y}(t_{n+1}; r) = \underline{y}(t_n; r) + \frac{1}{2}[K_1 + K_2]$$

where

$$\begin{aligned} K_1 &= hF[t_n, \underline{y}(t_n; r), \bar{y}(t_n; r)] \\ K_2 &= hF[t_{n+1}, \underline{y}(t_n; r), \bar{y}(t_n; r) + K_1] \end{aligned} \quad (21)$$

and

$$\bar{y}(t_{n+1}; r) = \bar{y}(t_n; r) + \frac{1}{2}[K_1 + K_2]$$

where

$$\begin{aligned} K_1 &= hG[t_n, \underline{y}(t_n; r), \bar{y}(t_n; r)] \\ K_2 &= hG[t_{n+1}, \underline{y}(t_n; r), \bar{y}(t_n; r) + K_1] \end{aligned} \quad (22)$$

Clearly,  $\underline{y}(t; r)$  and  $\bar{y}(t; r)$  converge to  $\underline{Y}(t; r)$  and  $\bar{Y}(t; r)$ , respectively

whenever  $h \rightarrow 0$  [4].

## 6 Numerical Results

In this section, the exact solutions and approximated solutions are obtained by Runge-kutta methods of order two are plotted in Figure 1, 2 and 3.

### Example 6.1

$$\begin{cases} y'(t) = t f(t), t \in [0,1] \\ y_1(0) = (1.5 + r\sqrt{e}, 5.5 - 0.5r\sqrt{e}) \end{cases}$$

The exact solution at  $t$  is given by

$$Y_1(t; r) = \left[ (1.5 + r\sqrt{e})e^{t^2/2}, (5.5 - 0.5r\sqrt{e})e^{t^2/2} \right], \quad 0 \leq r \leq 1$$

The exact solution at  $t = 0.1$  is given by

$$Y_1(0.1; r) = \left[ (1.5 + r\sqrt{e})e^{0.005}, (5.5 - 0.5r\sqrt{e})e^{0.005} \right], \quad 0 \leq r \leq 1$$

Using iterative solution, we have

$$\underline{y}_1(0; r) = 1.5 + r\sqrt{e}, \bar{y}_1(0; r) = 5.5 - 0.5r\sqrt{e}$$

**Table 1:** Exact solution

r	Trapezoidal fuzzy number
0	1.507519 , 5.527569
0.1	1.673217 , 5.444719
0.2	1.838916 , 5.361870
0.3	2.004614 , 5.279021
0.4	2.170313 , 5.196172
0.5	2.336011 , 5.113323
0.6	2.501710 , 5.030474
0.7	2.667408 , 4.947624
0.8	2.833107 , 4.864775
0.9	2.998805 , 4.781926
1	3.164504 , 4.699076



**Table 2:** Approximated solution & error for the different values of  $r$  &  $h$

Trapezoidal fuzzy number			
Runge-Kutta of order two			
$h$ $r$	0.1	0.01	0.001
Approximated solution			
0	1.688831 , 6.080374	1.525307 , 5.583395	1.510858 , 5.538890
0.1	1.874458 , 5.989239	1.692961 , 5.499709	1.676924 , 5.455871
0.2	2.060085 , 5.898105	1.860615 , 5.416023	1.842990 , 5.372850
0.3	2.245713 , 5.806970	2.028269 , 5.332337	2.009055 , 5.289834
0.4	2.431340 , 5.715835	2.195922 , 5.248651	2.175121 , 5.206814
0.5	2.616967 , 5.624700	2.363576 , 5.164965	2.341186 , 5.123796
0.6	2.802595 , 5.533566	2.531230 , 5.081279	2.507252 , 5.040777
0.7	2.988220 , 5.442430	2.698884 , 4.997593	2.673318 , 4.957758
0.8	3.173849 , 5.351295	2.866537 , 4.913907	2.839383 , 4.874739
0.9	3.359477 , 5.260161	3.034109 , 4.830221	3.005449 , 4.791720
1	3.545104 , 5.169026	3.201845 , 4.746535	5.171514 , 4.708701
Error for the different values of $r$ & $h$			
0	0.734118	0.073615	0.014661
0.1	0.745761	0.074733	0.014859
0.2	0.757404	0.075852	0.015055
0.3	0.769048	0.076971	0.015254
0.4	0.780691	0.078088	0.015450
0.5	0.792334	0.079207	0.015648
0.6	0.803977	0.080325	0.015845
0.7	0.815620	0.081444	0.016043
0.8	0.827263	0.082562	0.016240
0.9	0.838907	0.083681	0.016438
1	0.850550	0.084800	0.016635

### Graphical Representation of exact and approximated solution

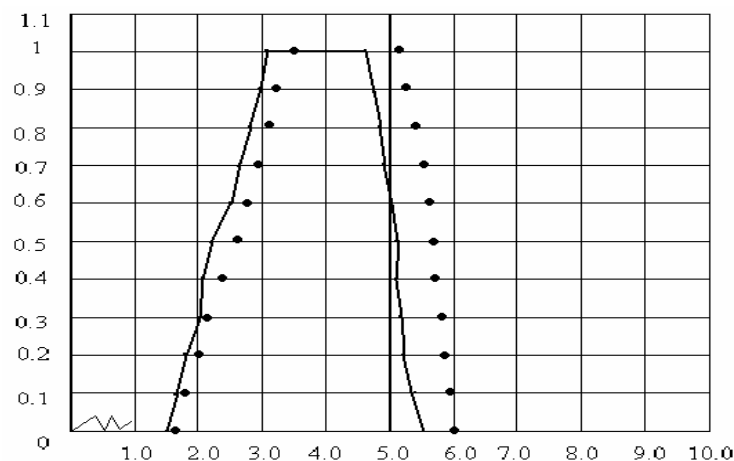


Figure 1 :  $h = 0.1$

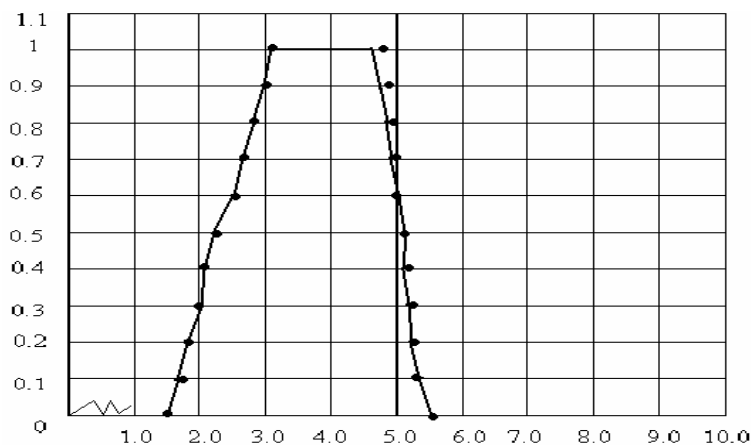


Figure 2 :  $h = 0.01$

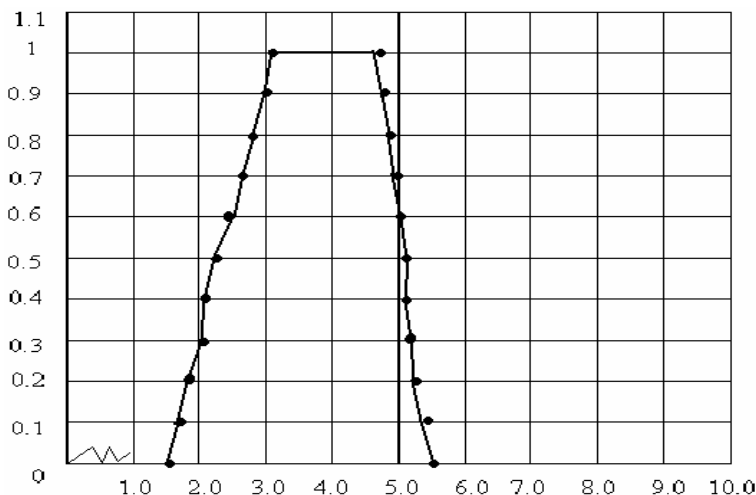


Figure 3 :  $h = 0.001$

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