Iterative Algorithm to Solve a System of Nonlinear Volterra-Fredholm Integral Equations Using Orthogonal Polynomials

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Abstract

This paper focuses on numerical method used to solve a linear and nonlinear system of Volterra or Fredholm integral equations using Legendre and Chebyshev collocation method. The method is based on replacement of the unknown function by a reduced series to length size by discarding the high degree terms of the Legendre or Chebyshev polynomials. This lead to a system of linear or nonlinear algebraic equations with Legendre or Chebyshev coefficients which will be solved using conjugate gradient or Newton iteration method respectively. Thus, by solving the matrix equation, Legendre or Chebyshev coefficients are obtained. Some numerical examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords: *System Volterra-Fredholm Integral equation, Legendre and Chebyshev polynomials, gradient method, Newton Method*

1 Introduction

Volterra and Fredholm integral equations arise in many problems pertaining to mathematical physics like heat conduction problems. They plays an important role in many branches of linear and nonlinear analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology [3]-[7]. In [12]-[16], the authors presented several method for solving linear and nonlinear Fredholm Integral equations. In [2, 1], chniti consider only the solution of a linear Fredholm or Volterra-Fredholm integral equation with singuler Kernel. There is no information what happen for a system of nonlinear Volterra-Fredholm Integral equation with singular kernel. To give a correct answer to this question, we will introduce a technique that can be generalized to the case of singuler Kernel. Here, we will consider the case of a nonlinear Fredholm integral equations of the type

$$\Phi(x) + \int_a^b \mathcal{K}(x,s)\mathcal{F}[s,\Phi(s)]ds = 0, \quad a \le x \le b,$$

where $\mathcal{K}(x,s)$ and $\mathcal{F}(x,s)$ are known functions, while $\Phi(x)$ is the unknown function. The function $\mathcal{K}(x,s)$ is known as the kernel of the Fredholm integral equation as play an important role for solving the problem and it's can be smooth or singular function. A. Hammerstein, who considered the case where $\mathcal{K}(x,s)$ is a symmetric and positive Fredholm kernel, i.e. all its eigenvalues are positive. If, in addition, the function $\mathcal{F}(x,s)$ is continuous and satisfies the condition

$$|\mathcal{F}(x,s)| \leq \beta_1|s| + \beta_2,$$

where β_1 and β_2 are positive constants and β_1 is smaller than the first eigenvalue of the kernel $K(x,s)$, the Hammerstein equation has at least one continuous solution. If, on the other hand, $\mathcal{F}(x,s)$ happens to be a nondecreasing function of s for any fixed x from the interval (a,b), Hammerstein's equation cannot have more than one solution. This property holds also if $\mathcal{F}(x,s)$ satisfies the condition

$$|\mathcal{F}(x,s_1) - \mathcal{F}(x,s_2)| \leq \beta|s_1 - s_2|,$$

where the positive constant β is smaller than the first eigenvalue of the kernel $\mathcal{K}(x,s)$.

Let us consider the following integral equation:

$$\Phi(t) = g(t) + \eta_1 \int_0^1 \mathcal{K}_1(t,s)F(\Phi(s))ds + \eta_2 \int_0^t \mathcal{K}_2(t,s)G(\Phi(s))ds, \tag{1}$$

where g, $\mathcal{K}_1$ and $\mathcal{K}_2$ are known functions and η_1, η_2 are two constants. and then we will give more general case, and we will solve a system of nonlinear Volterra-Fredholm integral equation of the form, for $i = 0, \ldots, n$:

$$\phi_i(s) = f_i(s) + \sum_{j=0}^{n} \int_{-1}^{s} K_{ij}(s,t)F(\phi_i(t))dt + \sum_{j=0}^{n} \int_{-1}^{1} K_{ij}(s,t)G(\phi_i(t))dt \tag{2}$$

where F and G two known nonlinear functions, f_i are known functions and ϕ_i are the unknown functions must be determined. The paper is organized as follows. In section 2, we present some results about Legendre polynomial. In section 3, we transform Fredholm integral equation to a system of algebraic equations. In section 4, some numerical examples are presented. We generalize our method to a nonlinear system 5, finally we conclude.

2 Legendre Polynomials

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common set of orthogonal polynomials is the Legendre polynomials. The Legendre polynomials P_n satisfy the recurrence formula:

$$\begin{aligned} (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x), n \in \mathbb{N} \\ P_0(x) &= 1 \\ P_1(x) &= x \end{aligned}$$

An important property of the Legendre polynomials is that they are orthogonal with respect to the L^2 inner product on the interval $[-1,1]$:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$$

where δ_{nm} denotes the Kronecker delta, equal to 1 if $m = n$ and to 0 otherwise. The polynomials form a complete set on the interval $[-1, 1]$, and any piecewise smooth function may be expanded in a series of the polynomials. The series will converge at each point to the usual mean of the right and left-hand limits. There are many application of the Legendre polynomial, for instance, when we solve Laplace's equation on a sphere, we want solutions that will be valid at the north and south poles (whose polar coordinates are $\cos\theta = \pm 1$), therefore, the physically meaningful solutions to Laplace's equation on a sphere are the polynomials of the first kind. The Legendre and Chebyshev polynomials are used also to solve several problems of differential equations or integral equations, for instance the Legendre pesudospectral method is used to solve the delay and the diffusion differential equations ([8], [9]). Other orthogonal polynomial like Chebyshev polynomials are used to introduce an efficient modification of homotopy perturbation00 method [10]. Also, the polynomial approximation is used to solve high order linear Fredholm integro differential equations with constant coefficient [13] and others ([11], [17]-[19]). The Legendre polynomial is used to approximate any continuous function. If we note by $\mathcal{P}$ the set all real-valued polynomials on $[-1, 1]$, then $\mathcal{P}$ is a dense subspace of the space of continuous function on the same interval equipped with the uniform norm, this is a particular case of the theorem of f the Stone-Weierstrass. basing on these properties of the Legendre polynomials we will solve the Fredholm integral equation.

3 System of algebraic equations

The Fredholm-Volterra integral equation (1) is considered in this paper. The expansion of the function Φ using infinite series of Legendre polynomials is given by:

$$\Phi(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t), \tag{3}$$

where $\alpha_n = (\Phi(t), P_n(t))$. If we truncate the infinite series in Equation(3) to only $N + 1$ terms, we obtain

$$\Phi(t) \simeq \sum_{n=0}^{N} \alpha_n P_n(t) = \alpha^T P(t), \tag{4}$$

where α and P are matrices given by

$$\alpha = [\alpha_0, \ldots, \alpha_n], \quad P = [P_0, \ldots, P_N]^T. \tag{5}$$

Substituting the truncated series (4) into Equation(1) we obtain

$$\alpha^T P(t) = g(t) + \eta_1 \int_0^1 \mathcal{K}_1(t, s) F(\alpha^T P(s)) ds + \eta_2 \int_0^t \mathcal{K}_2(t, s) G(\alpha^T P(s)) ds. \tag{6}$$

Using the Legendre collocation points defined by

$$t_i = \frac{i}{N},\ i = 0, 1, \ldots, N, \tag{7}$$

and substituting the variable t by t_i defined in (7) to get

$$\alpha^T P(t_j) = g(t_j) + \eta_1 \int_0^1 \mathcal{K}_1(t_j, s) F(\alpha^T P(s)) ds + \eta_2 \int_0^{t_j} \mathcal{K}_2(t_j, s) G(\alpha^T P(s)) ds. \tag{8}$$

The domain $[0,1]$ discretized into m equally spaced panels, or $N+1$ grid points, where the grid spacing is $h=\frac{1}{N}$, trapezoidal rules gives

$$\int_0^1 \mathcal{K}_1(t_j,s)F(\alpha^T P(s))ds \cong \frac{h}{2}(\mathcal{F}(s_0)+\mathcal{F}(s_N)+2\sum_{k=1}^{N-1}\mathcal{F}(s_k)), \tag{9}$$

where $\mathcal{F}(s)=\mathcal{K}_1(t_j,s)F(\alpha^T P(s))$, $h=\frac{1}{N}$, for any integer N, $s_i=ih,\ \ i=0,1,\ldots,N$ and

$$\int_0^{t_j} \mathcal{K}_2(t_j,s)G(\alpha^T P(s))ds \cong \frac{h_j}{2}(\mathcal{G}(\tilde{s}_0)+\mathcal{G}(\tilde{s}_N)+2\sum_{k=1}^{N-1}\mathcal{G}(\tilde{s}_k)), \tag{10}$$

where $\mathcal{G}(s)=\mathcal{K}_2(t_j,s)G(\alpha^T P(s))$, $h_j=\frac{t_j}{N}$, for an arbitrary integer N, $\tilde{s}_i=ih$. Now, we obtain the following system:

$$\alpha^T P(t_j)=g(t_j)+\eta_1\frac{h}{2}\left(\mathcal{F}(s_0)+\mathcal{F}(s_N)+2\sum_{k=1}^{N-1}\mathcal{F}(s_k)\right)+\eta_2\frac{h_j}{2}\left(\mathcal{G}(\tilde{s}_0)+\mathcal{G}(\tilde{s}_N)+2\sum_{k=1}^{N-1}\mathcal{G}(\tilde{s}_k)\right) \tag{11}$$

The system (11) is a $(N+1)$ linear or nonlinear algebraic equations, which can be solved for $\alpha_i\quad, i=0,1,\ldots,N$. Therefore the unknown function Φ can be determined. We recall that the conjugate gradient method is an algorithm for the numerical solution of particular systems of linear equations, it's implemented as an iterative algorithm used to solve a sparse systems that are too large to be handled by a direct implementation or other direct methods such as the Cholesky decomposition. Large sparse systems can be obtained when we solve a partial differential equations or optimization problems. In the other hand, the nonlinear system will be solved using Newton's method. A special code written using MATLAB is implemented for solving nonlinear equations using Newton's method. We note that Matlab has its routines for solving systems of nonlinear equations (one can look to "fsolve" in Matlab) which is based on Newton's method. The linear and nonlinear system will be solved using the conjugate gradient method and Newton's method respectively.

4 Examples

We confirm our theoretical discussion with numerical examples in order to achieve the validity, the accuracy. The computations, associated with the following examples, are performed by MATLAB 7.

Example 1. *Here, we will apply the technique presented in previous section to a linear integral equation, in order to show that the method presented can be applied. We consider the equation* (1) *with*

$$g(x)=x^3-(6-2e)e^x,\quad \eta_1=1,\quad \eta_2=1,$$
$$\mathcal{K}_1(x,y)=e^{(x+y)},\quad \mathcal{K}_2(x,y)=0,\quad F(\Phi(y))=\Phi(y),\ G(\Phi(y))=0.$$

The equation (1) *is as follows*

$$\Phi(x)=x^3-(6-2e)e^x+\int_0^1 e^{(y+x)}\Phi(y)dy. \tag{12}$$

The suggested method will be considered with $N = 4$*, and the approximate solution* $\Phi(x)$ *can be written in the following way*

$$\Phi_4(x) = \sum_{i=0}^{4} \alpha_i P_i(x) = \alpha^T P(x). \tag{13}$$

Using the same technique presented in previous section and using Equation(8) we obtain

$$\sum_{i=0}^{4} \alpha_i P_i(x_j) - (x_j^3 - (6-2e)e^{x_j}) - \frac{h}{2}(\mathcal{F}(y_0) + \mathcal{F}(y_m) + 2\sum_{k=1}^{m-1} \mathcal{F}(y_k)) = 0, \; j = 0,1,2,3,4, \tag{14}$$

where $\mathcal{F}(y) = e^{(y+x_j)} \sum_{i=0}^{4} \alpha_i P_i(y)$ *and the nodes* $y_{l+1} = y_l + h,\; l = 0,1,\ldots,N,\; y_0 = 0$ *and* $h = \frac{1}{N}$*. Equation(14) represents linear system of 5 algebraic equations in the coefficients* $\alpha_i, i = 0,\ldots,4$*, which will be solved by the conjugate gradient method and we get the following coefficients:*

$$\alpha_0 = -0.0048, \alpha_1 = 0.5955, \alpha_2 = -0.0015, \alpha_3 = 0.3998, \alpha_4 = -0.0001.$$

Hence, the approximate solution of Equation(13) is as follows:

$$\Phi(x) = -0.0048P_0(x) + 0.5955P_1(x) - 0.0015P_2(x) + 0.3998P_3(x) - 0.0001P_4(x).$$

corresponding to exact solution $\Phi(x) = x^3$.

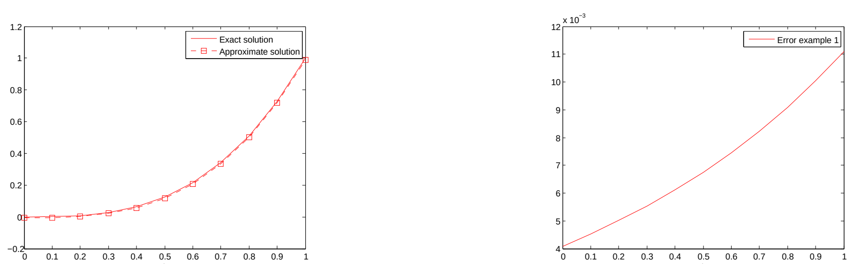

Figure 1: Error between exact solution and the present method with $N = 4$

In example 1, we have considered only $N = 4$ *terms in the expansion of the solution using Legendre polynomials, the Figure 1 gives the behavior of true (exact) and the approximate solution and the behavior of the error between them, we notice that the technique used is much more pertinent and can be considered as a profitable method to solve the linear integral equations.*

Example 2. *Unlike, the first example, here we will consider the case of a nonlinear integral equation with: Consider the equation (1) with the following functions and coefficients*

$$g(x) = 2xe^x - e^x + 1, \quad \eta_1 = 1, \quad \eta_2 = -1, \quad \mathcal{K}_1(x,y) = 0,$$

$$\mathcal{K}_2(x,y) = (x+y), \quad F(\Phi(y)) = 0, \quad G(\Phi(y)) = e^{\Phi(y)}.$$

Equation(1) takes the following form

$$\Phi(x) = 2xe^x - e^x + 1 - \int_0^t (y+x)e^{\Phi(y)}dy. \tag{15}$$

The suggested method will be considered with $N = 4$, and the approximate solution $\Phi(x)$ can be written in the following way

$$\Phi_N(x) = \sum_{i=0}^{4} \alpha_i P_i(x) = \alpha^T P(x). \tag{16}$$

The technique presented can be applied and using Eq.(8) we obtain

$$\sum_{i=0}^{4} \alpha_i P_i(x_j) - f(x_j) + \frac{h_j}{2}(\mathcal{F}(y_0) + \mathcal{F}(y_m) + 2\sum_{k=1}^{m-1} \mathcal{F}(y_k)) = 0, \quad j = 0,1,2,3,4, \tag{17}$$

where the nodes $y_{l+1} = y_l + h$, $l = 0,1,\ldots,m$, $y_0 = 0$ and $h_j = \frac{t_j}{m}$, $\mathcal{F}(y) = (y+x_j)e^{C^T P(y)}$. The Equation (17) presents nonlinear system of $N+1$ algebraic equations in the coefficients α_i. By solving it by using the Newton iteration method with suitable initial solution we obtain

$$\alpha_0 = 0.0002, \quad \alpha_1 = 0.9895, \quad \alpha_2 = 0.0022, \quad \alpha_3 = -0.0088, \quad \alpha_4 = 0.0023.$$

Hence, the approximate solution obtained from (16) written in he following way

$$\Phi(x) = 0.0002P_0(x) + 0.9895P_1(x) + 0.0022P_2(x) - 0.0088P_3(x) + 0.0023P_4(x).$$

associated to exact solution $\Phi(x) = x$. Figure 2 presents the behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution. From this Fig. 2 we notice that the proposed method can be considered as an pertinent method to solve the nonlinear integral equations.

Example 3. *Equation (1) will be considered with*

$$g(x) = xe + 1, \quad \eta_1 = -1, \quad \eta_2 = 1, \quad \mathcal{K}_1(x,y) = y + x,$$

$$\mathcal{K}_2(x,y) = 0, \quad F(\Phi(y)) = e^{\Phi(y)}, \quad G(\Phi(y)) = 0.$$

Equation(1) can be written as follows:

$$\Phi(x) = xe + 1 - \int_0^1 (y+x)e^{\Phi(y)}dy. \tag{18}$$

The exact solution associated to this problem the linear function $\Phi(x) = x$.

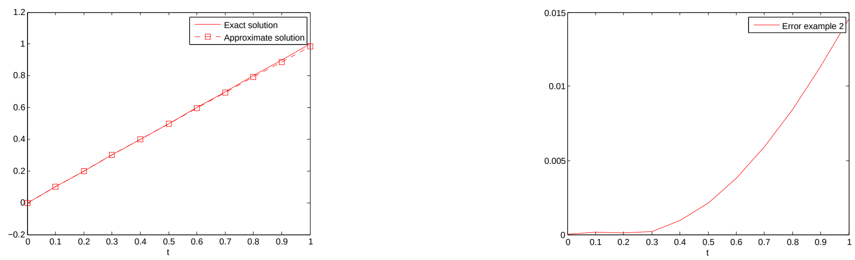

Figure 2: Error between exact solution and the present method with $N = 4$

The suggested method will be considered with $N = 3$, and the approximate solution $\Phi(x)$ can be written in the following way:

$$\sum_{i=0}^{3} \alpha_i P_i(x_j) - g(x_j) + \frac{h}{2}(\mathcal{F}(y_0) + \mathcal{F}(y_m) + 2\sum_{k=1}^{m-1} \mathcal{F}(y_k)) = 0, j = 0, 1, 2, 3, \tag{19}$$

where $y_{l+1} = y_l + h,\ l = 0, 1, \ldots, N,\ y_0 = 0$ and $h = \frac{1}{N}$ and $\mathcal{F}(y) = (y + x_j).e^{(\sum_{i=0}^{3} \alpha_i P_i(y))}$. We have a system of nonlinear algebraic equations (19). The solution can be obtained using Newton iteration method with suitable initial solution, and one can obtain:

$$\alpha_0 = -0.0023, \alpha_1 = 1.0013, \alpha_2 = 0.0,\ \alpha_3 = 0.0\,.$$

The approximated solution derived in this example is:

$$\Phi(x) = -0.0023P_0(x) + 1.0013P_1(x) + 0.0P_2(x) + 0.0P_3(x).$$

Figure 3 gives the behavior of the approximate solution using the proposed method with $N = 3$ and the exact solution.

Example 4. *We consider here a nonlinear integral equation with the following function and coefficients:*

$$g(x) = \frac{x}{2} - \frac{x^4}{12} - \frac{1}{3},\ \ \eta_1 = 1,\ \ \eta_2 = -1,\ \ \mathcal{K}_1(x, y) = y + x,$$
$$\mathcal{K}_2(x, y) = y - x,\ \ F(\Phi(y)) = \Phi(y),\ \ G(\Phi(y)) = \Phi^2(y).$$

Equation(1) becomes

$$\Phi(x) = \frac{x}{2} - \frac{x^4}{12} - \frac{1}{3} + \int_0^1 (y + x)\Phi(y)dy + \int_0^t (y - x)\Phi^2(y)dy. \tag{20}$$

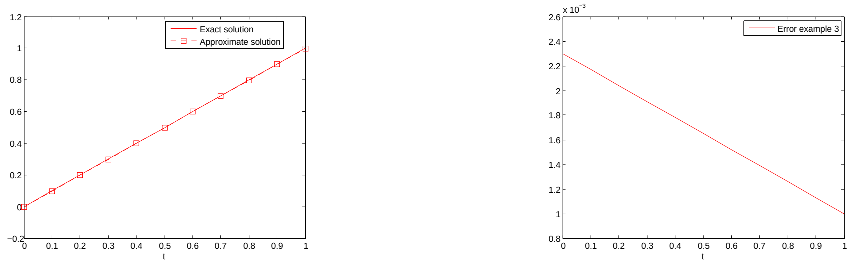

Figure 3: Error between exact solution and the present method with $N = 3$

The technique presented can be applied using only the first five Legendre Polynomials function and the approximate solution can be written as:

$$\Phi_N(x) = \sum_{i=0}^{4} \alpha_i P_i(x) = \alpha^T P(x). \tag{21}$$

The same technique and using equation (8) *we get*

$$\sum_{i=0}^{4} \alpha_i P_i(x_j) - g(x_j) - \frac{h}{2}(\mathcal{F}(\bar{y}_0) + \mathcal{F}(\bar{y}_N) \quad +2\textstyle\sum_{k=1}^{N-1} \mathcal{F}(\bar{y}_k)) - \frac{h_j}{2}(\mathcal{G}(y_0) + \mathcal{G}(y_N)$$
$$+2\textstyle\sum_{k=1}^{N-1} \mathcal{G}(y_k)) = 0, \tag{22}$$

where $\bar{y}_{l+1} = \bar{y}_l + h$, $y_{l+1} = y_l + h_j, l = 0, 1, \ldots, N$, $s_0 = \bar{y}_0 = 0$ $h = \frac{1}{N}$, $h_j = \frac{x_j}{N}$, *and* $\mathcal{F}(y) = (y + x_j)(\sum_{i=0}^{4} \alpha_i P_i(y))$, $\mathcal{G}(y) = (x_j - y)(\alpha^T P(y))^2$. *Equation* (22) *is a nonlinear system of* 5 *algebraic equations. The Newton iteration method can be used and we obtain the following coefficients:*

$$\alpha_0 = -0.0012, \quad \alpha_1 = 0.9987, \quad \alpha_2 = -0.0039, \quad \alpha_3 = 0.0007, \quad \alpha_4 = -0.0017.$$

Hence, the approximate solution can be written using the first five Legendre polynomial as:

$$\Phi(x) = -0.0012P_0(x) + 0.9987P_1(x) - 0.0039P_2(x) + 0.0007P_3(x) - 0.0017P_4(x).$$

and it's will be compared to the exact solution which is $\Phi(x) = x$. *Figure* (4) *presents the behavior of the approximate solution and the exact solution, we notice that the proposed technique can be well chosen as a pertinent way to solve the nonlinear Volterra Fredholm integral equations.*

The figure 4 and Table (1) *show the behavior of the exact solution and the approximate solution using* $N = 4$.

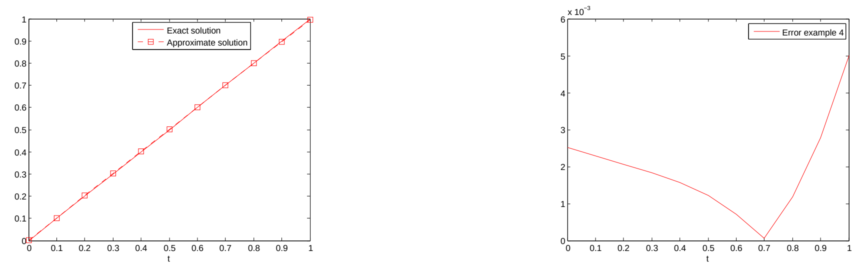

Figure 4: Error between exact solution and the present method with $N = 4$

5 System of Nonlinear Volterra-Fredholm integral equations

In this section, we will develop a general method to solve a system of nonlinear Volterra-Fredholm integral equation system by using the Chebyshev polynomials instead of Legendre polynomials used in the first part, but the computation it's the same. We will solve the system of $n + 1$ nonlinear Volterra-Fredholm integral equations are defined as follows: for $i = 0, \ldots, n$:

$$\phi_i(s) = f_i(s) + \sum_{j=0}^{n} \int_{-1}^{s} K_{ij}(s,t)F(\phi_i(t))dt + \sum_{j=0}^{n} \int_{-1}^{1} K_{ij}(s,t)G(\phi_i(t))dt \tag{23}$$

where F and G two known nonlinear functions, f_i are known functions and ϕ_i are the unknown functions must be determined. The system (23) introduce a $n+1$ Volterra-Fredholm integral equation with $n + 1$ unknowns. We use Chebyshev polynomials for solving nonlinear Volterra-Fredholm integral equations, one can propose other type of polynomials to solve the same problem using the same technique.

The Chebyshev polynomials of the first kind can be defined as the unique polynomials satisfying

$$T_n(\cos(\vartheta)) = \cos(n\vartheta), n = 0, 1, \ldots,$$

Then the inner product is given by

$$< T_i, T_j >= \int_{-1}^{1} T_i(x)T_j(x)w(x)dx \tag{24}$$

where $w(x) = (1 - x^2)^{-1/2}$ With respect to the inner product which is defined in Chebyshev polynomials are orthogonal

$$< T_i, T_j >= \pi\delta_{ij} \tag{25}$$

t	Exact solution	Approximate solution	Error
0	0	0.0025	0.0025
0.1	0.1	0.1023	0.0023
0.2	0.2	0.2021	0.0021
0.3	0.3	0.3018	0.0018
0.4	0.4	0.4016	0.0016
0.5	0.5	0.5012	0.0012
0.6	0.6	0.6007	0.0007
0.7	0.7	0.6999	0.0001
0.8	0.8	0.7988	0.0012
0.9	0.9	0.8972	0.0028
1	1	0.9950	0.0050

Table 1: Example 4. Comparison between exact and approximate solution

where δ_{ij} denote the Kronecker's delta. We will use Chebyshev polynomials to approximate the unknown function, they are used as a collocation basis to solve system of nonlinear Volterra-Fredholm integral equation and reduce it to a linear or nonlinear system of algebraic equations. Newton's iterative method can be used for solving nonlinear algebraic system. If f defined on $[-1,1]$, then we have

$$f(s) = \sum_{i=0}^{\infty} c_i T_i(s)$$

and after truncation we have

$$f(s) = \sum_{i=0}^{N} c_i T_i(s) = C^t T(s)$$

where

$$C = [c_0, c_1, c_2, \ldots, c_N]^t, \quad T(s) = [T_0(s), T_1(s), T_2(s), \ldots, T_N(s)]^t$$

where the coefficient c_i are defined as

$$c_i = \begin{cases} \frac{1}{\pi}\int_{-1}^{1}(1-s^2)^{-1/2} f(s)ds & \text{if } i = 0 \\ \frac{2}{\pi}\int_{-1}^{1}(1-s^2)^{-1/2} f(s)T_i(s)ds & \text{if } i > 0 \end{cases}$$

For simplicity we will consider some special cases where $F(\phi_i) = \phi_i^p$ and $G(\phi_i) = \phi_i^q$ where p, q two integer. In this case, we have

$$F(\phi_i(s)) = \phi_i^p(s) = \tilde{C}_p^t T(s)$$

where $\tilde{C}^t$ can be obtained from the vector C. For example, in the case where $p = 2$ we have (Thanks to the software Maple for helping us to do the computation)

$$\tilde{C}_2 = \frac{1}{2}\begin{pmatrix} 2c_0^2 + c_1^2 + c_2^2 + c_3^2 \\ 4c_0c_1 + 2c_1c_2 + 2c_2c_3 \\ c_1^2 + 4c_0c_2 + c_1c_3 \\ 2c_1c_2 + 4c_0c_3 \end{pmatrix}$$

The kernel

$$K(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N} T_i(s)K_{ij}T_j(t)$$

where

$$K_{ij} =< T_i(s), < K(s,t), T_j(t) >>$$

So K *becomes a* $(N+1)\times(N+1)$ *matrix with elements* K_{ij} *Therefore*

$$K(s,t) = T^t(s)K(s,t)T(t)$$

For $i, j = 0, \ldots, N$, *we have the following relation:*

$$\phi_i(s) = T^t(s)\Phi_i \implies (\phi_i(s))^p = T^t(s)\tilde{\Phi}_{ip}$$

and

$$K_{ij} = T^t(s)K_{ij}T(t)$$

where

$$\tilde{F}_{ip} = [f_{i0}, f_{i1}, f_{i2}, \ldots, f_{iN}]^t$$

Some special formula will be used in our computation:

$$\int_{-1}^{x} T_{n-1}(t)dt = \frac{1}{2n}T_n(x) - \frac{1}{2(n-1)}T_{n-2}(x) + \frac{(-1)^{n-1}}{1-(n-1)^2}T_0(x), \quad n \geqslant 3$$

$$\int_{-1}^{x} T_0(t)dt = T_0(x) + T_1(x)$$

$$\int_{-1}^{x} T_1(t)dt = \frac{1}{4}\left(-T_0(x) + T_2(x)\right)$$

these integral can be written using a matrix form P:

$$\int_{-1}^{x} T(t)dt = PT(x), \ \int_{-1}^{1} T(t)dt = PT(1)$$

where P *is a* $(n+1)\times(n+1)$ *matrix defined as follows:*

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \ldots & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ \frac{(-1)^{n-1}}{1-(n-1)^2} & 0 & 0 & 0 & \ldots & 0 & \frac{1}{2n} \\ \frac{(-1)^n}{1-n^2} & 0 & 0 & 0 & & \frac{-1}{2(n-1)} & 0 \end{pmatrix}$$

using Chebyshev Polynomial we have the following relation:

$$T(s)T(s)^t C = \mathbf{C}^t T(s)$$

where $\mathbf{C}$ *is a* $(n+1)\times(n+1)$ *square matrix given by*

$$\mathbf{C} = \frac{1}{2}\begin{pmatrix} 2c_0 & c_1 & \dots & c_i & \dots & c_{n-1} & c_n \\ 2c_1 & 2c_0+c_2 & \dots & c_{k-1}+c_{k+1} & \dots & c_{n-2}+c_n & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 2c_{n-1} & c_{n-2}+c_n & \dots & c_{n-k-1} & \dots & 2c_0 & c_1 \\ 2c_n & c_{n-1} & \dots & c_{n-i} & \dots & c_1 & 2c_0 \end{pmatrix}$$

Now, the system of Volterra-Fredholm integral equation becomes:

$$T^t(x)\tilde{\Phi}_{ip} = f_i(x) + \sum_{j=0}^{n}\int_{-1}^{x} T^t(x)K_{ij}T^t(t)\tilde{\phi}_{ip_j}dt + \sum_{j=0}^{n}\int_{-1}^{1} T^t(x)K_{ij}T(t)T^t(t)\tilde{\phi}_{iq_j}dt$$

this lead to the following system: for $i = 0,1,2,\dots,n$ *we have*

$$f_i(x) = T^t(x)\tilde{\Phi}_i - \sum_{j=0}^{n} T^t(x)K_{ij}\tilde{\phi}_{ip_j}P^t(x) - \sum_{j=0}^{n} T^t(x)K_{ij}\tilde{\phi}_{ip_j}P^t(1) \tag{26}$$

the equation 26 can be evaluated at the collocation points $\{x_k\}, k = 0,\dots,N$ *in the interval* $[-1,1]$ *then we get a system of* $(n+1)\times(N+1)$ *equation with* $(n+1)\times(N+1)$ *unknowns. For* $k = 0,\dots,N$ *and* $i = 0,\dots,n$ *we are going to solve:*

$$f_i(x_k) = T^t(x_k)\tilde{\Phi}_i - \sum_{j=0}^{n} T^t(x_k)K_{ij}\tilde{\phi}_{ip_j}P^t(x_k) - \sum_{j=0}^{n} T^t(x_k)K_{ij}\tilde{\phi}_{ip_j}P^t(1) \tag{27}$$

The relation 27 leads to a linear or nonlinear system of equations such that the unknown coefficients can be found. In the last of this paper and to confirm the strategy proposed we present some numerical example. We use Newton's iterative method for solving the generated nonlinear system, Maple and Matlab are used in our case to do our numerical test.

Consider the following nonlinear integral equations system:

$$\begin{aligned} \phi_1(x) &= f_1(x) + \int_{-1}^{x} x^2t^3\phi_1^4(t)dt + \int_{-1}^{1}(3t-x^2)\phi_2^7(t)dt \\ \phi_2(x) &= f_2(x) + \int_{-1}^{x} xt\phi_1^2(t)dt + \int_{-1}^{1} 3t^2x\phi_2^5(t)dt \end{aligned} \tag{28}$$

where

$$\begin{aligned} f_1(x) &= 2\,x - \frac{64511}{2145} - 2\,x^{10} + \frac{2047181}{180180}\,x^2 - \frac{32}{7}\,x^9 - 4\,x^8 - 8/5\,x^7 - 1/4\,x^6 \\ f_2(x) &= x^2 - \frac{6935}{858}\,x - x^5 - 4/3\,x^4 - 1/2\,x^3 \end{aligned}$$

and the exact solution are $\phi_1(x) = 2x+1, \phi_2(x) = x^2 + x$

Table 2 gives a comparison between the exact and approximation solution.

x_i	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
ϕ_1	-1	-0.5	0	0.5	1	1.5	2	2.5	3
ϕ_1^9	-0.987	-0.495	0.001	0.476	1.01	1.4892	1.997	2.4964	2.9987
$\lvert\text{Error}_1\rvert$	0.013	0.005	0.001	0.024	0.01	0.0108	0.0030	0.0036	0.0013
ϕ_2	0	-0.1875	-0.25	-0.1875	0	0.3125	0.7500	1.3125	2
ϕ_2^9	0.001	-0.1864	-0.2499	-0.1869	0.001	0.3114	0.7489	1.3114	1.99876
$\lvert\text{Error}_2\rvert$	0.0010	0.0011	0.0001	0.0006	0.0010	0.0011	0.0011	0.0011	0.0012

Table 2: Comparison between the approximate solution and exact solution

6 Conclusion

We solved a system of Volterra-Fredholm integral equations by using Chebyshev collocation method. The properties of Chebyshev or Legendre polynomials are used to reduce the system of Volterra Fredholm integral equations to a system of nonlinear algebraic equations. The method presented in this paper based on the Legendre and Chebyshev polynomials is suggested to find the numerical solution which will be compared to the analytic solution. The iterative method conjugate gradient method and Newton's method are used to solve the linear and nonlinear system. Analyzing the numerical solution and the exact solution declare that the technique used is very effective and convenient.The approach used is tested with different examples to show that the accuracy improves with increasing N. Moreover, using the obtained numerical solution, we can affirm that the proposed method gives the solution in an great accordance with the analytic solution. In addition, one can investigate other type of a nonlinear Fredholm integro differential equation with singular kernel.

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