Small-Singular Submodules and SY-Extending Modules

Mehdi Sadiq Abbas [#1], **Faten Hashim Mohammed** [#2]
Department of Mathematics, College of Science, University of Mustansiriyah, Baghdad, Iraq.

ABSTRACT

In this work, we introduce and study the concept of small-singular submodules as a generalization of the singular submodules. A number of properties and characterization of this concept are obtained. Also we introduce small-closure of arbitrary submodules and small related submodules, as well as we introduce and study the concept of small y-extending module as a generalization of the y-extending module consequent a generalization of extending module. More than that we introduce a small y-extending modules which is generalization of y-extending modules.

Key words: Essential submodules, small submodules, s-essential submodules, y-closed submodules, sy-closed submodules, (s-) singular module, (s-) closure submodules, (s-) related submodules, y-extending and sy- extending modules.

1. INTRODUCTION

In this paper, R an associative ring with identity, and M a unitary right R-module. It is well known that a submodule N of an R-module M is said to be small in M notationally, $N \leq_s M$, if $N + L = M$ for every submodule L of M, then L = M. Dually, a nonzero submodule N of M is essential, if whenever $N \cap L = (0)$, then L = (0) for every submodule L of M. In this case, we write $N \leq_e M$ and M is called essential extension of N [7]. The concept of essential submodule has been generalized to small-essential submodule by D. X. Zhou and X. R. Zhang, where it is defined by them as follows: Let N be a submodule of an R-module M. N is said to be small-essential in M (denoted by $N \leq_{se} M$), if $N \cap L = 0$ with $L \leq_s M$ implies $L = 0$[12].

Goldie [5], Johnson and Wong [6], defined the closure of a submodule N of an R-module M (denoted by cl(N)), as follows cl(N) = { m ∈ M | [N:M] is an essential right ideal of R}. Equivalently, cl(N) = { m∈M | mI ⊆ N} for some essential ideal I of R}. Where [N:M] the residual of M in N defined as follows: [N: M] = { r ∈ R | rM ⊆ N }[9]. In particular if N = 0, then cl(0) is the singular submodule and denoted by Z(M) where Z(M) = {m ∈ M : $r_R(m) \leq_e R$} [4]. Moreover, if Z(M) = 0, then M is called a nonsingular R-module and s-singular if $Z^s(M) = M$. In this paper, we define the small closure of N (denoted by scl(N)), it is stronger than the concept of closure submodules. In particular if N = 0, then scl(0) is the small-singular R-module and denoted by $Z^s(M)$. Moreover, if $Z^s(M) = 0$, then M is called a small-nonsingular module and small-singular module if $Z^s(M) = M$. And we give the definition of small related of two submodules (denoted by $\sim^s$) which is generalization the concept of related [8].

A. Tercan [11] introduced the concept of "CLS-modules" as a generalization of extending modules. We introduce the small y-extending (shortly sy-extending) modules as a generalization of y-extending modules (CLS). An R-module M is called sy-extending, if every sy-closed submodule is a direct summand. Where N is sy-closed submodule of M if M\N is s-nonsingular. It is stronger than the concept of y-closed submodules [4]. Also we study the relationships between sy-closed submodules, s-closed submodules [1] and y-closed submodules.

2. Small-Singular Submodules

In this section we will give definition for the small-singular which depends on s-essential ideal and small closure with some of their properties.

Definition (1.1): Let M be an R-module, for each submodule N of M, we define

scl(N) = { x ∈ M | xI ⊆ N for some s-essential right ideal I of R}

Equivalently, $scl(N) = \{ x \in M \mid [N: x] \leq_{se} R\}$. It is clear that $N \subseteq cl(N) \subseteq scl(N)$. We call scl(N) the small closure of N.

In particular, we define the small singular (shortly s-singular) of M (denoted by $Z^s(M)$) $Z^s(M) = \{ x \in M \mid ann(x) \leq_{se} R \}$ and equivalently $Z^s(M) = \{ x \in M \mid xI = 0$ for some s-essential right ideal I of R}, it is clear that $scl(0) = Z^s(M)$ and define scl(scl(0)) the second s-singular of M, denoted by $Z_2^s(M)$. If $Z^s(M) = 0$, then M is called an s-nonsingular module and s-singular module if $Z^s(M) = M$. Note that in case R is right hollow ring (i.e. every proper right ideal in R is small) and M is R-modules, then $Z(M) = Z^s(M)$.

Proposition (1.2): Let M be an R-module and N is a s-essential submodule in M. Then [N: M] is a s-essential right ideal of R.

Proof: Clear by by [12, Pro.2.7].

Remarks and Examples (1.3):

1. scl(N) is a submodule of M.

Proof: It is clear that scl(N) is non-empty. Let x, y be two elements in scl(N). Then there are two s-essential right ideals I and J such that $xI \subseteq N$ and $yJ \subseteq N$ by [12] we have that $I \cap J$ is s-essential in R, therefore $(x + y)(I \cap J) \subseteq N$, this implies that $x + y \in scl(N)$. For each $r \in R$ and $x \in scl(N)$, we have by above proposition, $[I{:}\, r] \leq_{se} R$ so $(xr)[I{:}r] \subseteq xI \subseteq N$ whence $xr \in scl(N)$. Thus scl(N) is a submodule of M.

2. Every singular submodule is s-singular. But the converse may not true, for example: Z_6 as Z_6-module then. $Z^s(2Z_6) = 2Z_6$ but $Z(2Z_6) = 0$, because the essential ideal of Z_6 only Z_6 but s-essential ideal of Z_6 are $\{ Z_6, 2Z_6, 3Z_6 \}$.

3. Every s-nonsingular submodule of M is nonsingular. The converse may not true clarify in (2).

The following two propositions give some properties of s-singular submodules:

Proposition (2.4): Let M be an R-module. Then the following hold:

1. If $f: M \to N$ is a R-homomorphism then $f(Z^s(M)) \subseteq Z^s(N)$. In particular, $Z^s(M)$ is fully inverant submodule in M.

2. If N is a submodule of M, then $Z^s(N) = N \cap Z^s(M)$.

3. If N is a submodule of M, then $Z_2^s(N) = N \cap Z_2^s(M)$.

4. $M Z^s(R) \subseteq Z^s(M)$.

5. $M Z_2^s(R) \subseteq Z_2^s(M)$.

Proof:

1. Let $w \in f(Z^s(M))$ then there exist $m \in Z^s(M)$ such that $w = f(m)$ and for each $I \leq_{se} R$ then $mI = 0$. We claim that $wI = 0$, $wI = f(m)I = f(mI) = f(0) = 0$, thus $w \in Z^s(N)$.

2. And (3) directly from the definition.

4. Consider the following map $\varphi_m : R_R \to M_R$ such that $\varphi_m(r) = mr$ for each $r \in R$ and $m \in M$ and φ_m is homomorphism, thus $mZ^s(R) = \varphi_m(Z^s(R)) \subseteq Z^s(M)$.

5. By the same way in (4).

Recall that a monomorphism f: M⟶N is s-essential in case Imf $\leq_{se}$ N [12].

Proposition (2.5): (**a**) An R-module C is s-nonsingular if and only if $\mathrm{Hom}_R(A, C) = 0$ for all s-singular modules A.

(**b**) A finitely generated R-module C is s-singular if and only if there exist a short exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ such that f is s-essential monomorphism.

Proof : (a) If A is s-singular, C is s-nonsingular, and f: A → C, then $f(A) = f(Z^s(A)) \subseteq Z^s(C) = 0$, then f (A) = 0. Thus $\mathrm{Hom}_R(A, C) = 0$.

Conversely, if $\mathrm{Hom}_R(A, C) = 0$ for all s-singular modules A, then in particular $\mathrm{Hom}_R(Z^s(C), C) = 0$. Now the inclusion map i: $Z^s(C) \to C$ is zero, hence $Z^s(C) = 0$. □

(b) Assume that C is a finitely generated s-singular and choose a short exact sequence,

$0 \to A = \ker(g) \overset{i}{\to} B \overset{g}{\to} C \to 0$ such that B is finitely generated free module. Let $\{b_i\}_{i=1}^n$ is a basis for B , then for each i = 1, … , n , g (b_i) such that C = $Z^s(C)$ there exist s-essential right ideal I_i of R such that g (b_i) I_i =0, then g $(b_iI_i) = 0$ hence $b_iI_i \subseteq \ker g = \mathrm{Im}\, i = A$. Since $I_i \leq_{se}$ R for each i = 1, … , n, we get $b_iI_i \leq_{se} b_iR$ for each i = 1, … , n, since suppose for each i=1, …, n($\neq 0$) $b_ix \in b_iR$, with $b_ixR \leq_s b_iR$, then $(\neq 0)x \in R$ and $xR \leq_s R$ and since $I_i \leq_{se} R$, then there is an element $r \in R$ such that $(\neq 0)xr \in I_i$ and by uniqness of basis we get $(\neq 0)b_ixr \in b_iI_i$. Hence by [12, Pro.2.7] $\bigoplus_{i=1}^n b_i I_i \leq_{se} \bigoplus_{i=1}^n b_i R_R = B$. Inasmuch as $\bigoplus_{i=1}^n b_i I_i \leq A$, we obtain A $\leq_{se}$ B, and the inclusion map A = ker(g) ⟶ B is a s-essential monomorphison.

Conversely, first assume that we have an exact sequence. Now suffices to show that $C \leq Z^s(C)$. let c ∈ C and given any B there exist $r_1, r_2, \dots, r_n \in R$ such that $b = \sum_{i=1}^n b_i r_i$ and g (b) = c. when $\{b_i\}_{i=1}^n$ is a basis for B. Define φ: R → B by φ(r)= br, φ is R-homomorphism by hypothesis f(A) $\leq_{se}$ B then by [12, Pro.2.7], φ^{-1} (f (A)) $\leq_{se}$ R, that is, the right ideal I = {r ∈ R | br ∈ f(A)} $\leq_{se}$R. Now bI ≤ f (A) = ker(g) by exact sequence which implies that g(bI) = 0, hence (g(b))I = 0 then c = g(b) ∈ $Z^s(C)$. Therefore C = g(B) ⊆ $Z^s(C)$, since g is onto, hence C = $Z^s(C)$ and C is s-singular. □

The following proposition characterizes the small essentially in terms of small singularity.

Proposition (2.6): Let A be a submodule of s-nonsingular module B. Then B/A is s-singular if and only if A $\leq_{se}$ B.

Proof: Suppose that B /A is s-singular. Let x($\neq 0$) ∈ B with xR is small in B. Then $\bar{x} = x + A \in B/A$. Now since B/A is s-singular, then there exist I $\leq_{se}$ R with $\bar{x}$ I = A then x I + A = A, hence xI ⊆ A and B is s-nonsingular then $x \notin Z^s(B)$, then x I ≠ 0 and 0 ≠ x I = xI ∩ A ⊆ xR ∩ A so xR ∩ A ≠ 0. Then 0 ≠ x R ⊆ A. Therefore, A $\leq_{se}$ B. Conversely; let A $\leq_{se}$B and consider the following exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{g} B/A \to 0$ and since i is s-essential monomorphism then by (Pro.(2.5)(b)) B/A is s-singular. □

Remarks and Examples (2.7):

1. A submodules of s-singular (s-nonsingular) R-module are s-singular (s-nonsingular).

Proof: let $A \leq B$ and B is s-singular, then $Z^s(A) = A \cap Z^s(B) = A$ and so A is s-singular.

2. Let A be s-nonsingular R-module. Then every s-essential extension B of A with $Z^s(B)$ small in B is s-nonsingular.

Proof: Let A is s-nonsingular, then since $A \cap Z^s(B) = Z^s(A) = 0$ and by assumption $Z^s(B) \leq_s B$. We must have $Z^s(B) = 0$, since $A \leq_{se} B$ then B is s-nonsingular .

3. Every essential extension B of s-nonsingular submodule is s-nonsingular. (as a bove without external condition)

4. If $\{ C_\alpha | \alpha \in \Lambda \}$ is a collection of s-nonsingular R-module $C_\alpha, \alpha \in \Lambda$, then $\prod_{\alpha \in \Lambda} C_\alpha$ is s-nonsingular.

Proof: If $\{C_\alpha\}$ is any collection of s-nonsingular modules and A is s-singular then have $Hom(A, C_\alpha) = 0$ for all α by Pro.(2.5)(a) and by [7, P.87], whence $Hom(A, \prod_{\alpha \in \Lambda} C_\alpha) \cong \prod_{\alpha \in \Lambda} Hom(A, C_\alpha) = \prod_{\alpha \in \Lambda}(0) = 0$ so that $\prod_{\alpha \in \Lambda} C_\alpha$ is s-nonsingular.

5. If $A \leq B$ and B is s-singular module, then B/A is s-singular module.

Proof: The projection map $B \to \frac{B}{A}$ must carry $Z^s(B) \to Z^s(\frac{B}{A})$, then $\frac{B}{A} = \frac{Z^s(B)}{A} \leq Z^s(\frac{B}{A})$ and so $\frac{B}{A}$ is s-singular.

6. The finite direct sum of s-singular modules is s-singular.

Proof: Let $\{C_i\}_{i=1}^n$ be any collection of s-singular modules then by Pro.(2.5)(b), gives us a short exact sequence $0 \to A_i \to B_i \to C_i \to 0$ such

that $A_i \to B_i$ is s-essential monomorphisom for each i = 1, ... n.

Now $0 \to \bigoplus_{i=1}^n A_i \to \bigoplus_{i=1}^n B_i \to \bigoplus_{i=1}^n C_i \to 0$ is exact too. And by [12, Pro.2.7] says that $\bigoplus_{i=1}^n A_i \to \bigoplus_{i=1}^n B_i$ is s-essential monomorphism. Hence by Pro.(2.5)(b), we say that $\bigoplus_{i=1}^n C_i$ is s-singular.

7. The module extension of s-nonsingular R-module is s-nonsingular.

Proof: Suppose that $0 \to C \to B \to A \to 0$ is an exact sequence of modules with C, A s-nonsingular. A ccording to pro.(2.5)(a) we have $Hom_R(M,C) = 0$ and $Hom_R(M,A) = 0$ for any s-singular module M. By exactness of the sequence $0 \to Hom_R(M, C) \to Hom_R(M, B) \to Hom_R(M, A)$. We obtain $Hom_R(M, B) = 0$ and by Pro.(2.2.6)(a) show that B is s-nonsingular.

8. In s-nonsingular modules, every essential extension and module extensions of s-nonsingular are s-nonsingular (see (3),(7)), but we cannot conclude that the s-singular modules are closed under either module extensions or essential extensions. For example, let Z_4 as Z_4-module if the submodules of Z_4 are 0, $2Z_4$ and Z_4 , since every nonzero submodule of Z_4 contains $2Z_4$ we obtain the s-essential $\{2Z_4 , Z_4\}$. Now $2Z_4 . 2Z_4 = 0$, hence $2Z_4 \leq Z^s(Z_4)$. Since $1 \notin Z^s(Z_4)$, it follows that $Z^s(Z_4) = 2Z_4$. Now $2Z_4$ is s-singular R-module and since $Z_4/2Z_4 \cong 2Z_4$, $Z_4/2Z_4$ is s-singular thus Z_4 is an extension of the s-singular module $2Z_4$ by the s-singular module $Z_4/2Z_4$, yet Z_4 is not s-singular. We also note that Z_4 is an essential extension of the s-singular module $2Z_4$. Therefore the class of all s-singular R-modules is not closed under either module extensions or essential extensions.

3. Small-Related Submodules

Definition (3.1): Let N_1 and N_2 be submodules of M. We say that N_1 and N_2 are small-related (denoted by $N_1 \sim^s N_2$) provided that $N_1 \cap X = 0$ if and only if $N_2 \cap X = 0$, where X is small submodule of M.

If $N_1 \subseteq N_2$ then $N_1 \sim^s N_2$ simply gives $N_1 \leq_{se} N_2$.

Lemma (3.2): Let L and N be submodules of an R-module M, then.

(i) $N + scl(0) \sim^s scl(N)$;

(ii) $L \sim^s N$ implies that $L \subseteq scl(N)$;

(iii) $scl(N) \sim^s scl\, scl(N)$.

Proof:(i) Let X be a small submodule of an R-module M such that $X \cap (N + scl(0)) = 0$. For any $x \in X \cap scl(N)$, there is a right ideal $I \leq_{se} R$ such that $xI \subseteq N$. Then $xI \subseteq X \cap N = 0$, implies that $x \in X \cap scl(0) = 0$ and hence $x = 0$. And the converse is clear.

(ii) Let $l \in L$ and define a homomorphism $\alpha: R \longrightarrow M$ by $\alpha(r) = lr$ for each $r \in R$. Since $L \leq_{se} M$ so by [12, Pro.2.7] we get $I = \{ r \in R \mid lr \in N\}$ is s-essential right ideal of R and hence $l \in scl(N)$.

(iii)Replacing N by scl(N) in (i) we get $scl\, scl(N) \sim^s (scl(N) + scl(0)) = scl(N)$ (i.e., $scl(N) \leq_{se} scl\, scl(N)$) .□

Proposition (3.3): Every submodule of s-nonsingular module is s-essential in its s-closure.

Proof: Let M be s-nonsingular R-module and N a submodule of M. Since $N + scl(0) \sim^s scl(N)$, i.e. $N + scl(0) \leq_{se} scl(N)$ and $scl(0) = Z^s(M) = 0$, so $N \leq_{se} scl(N)$. □

Definition (3.4): Let M be an R-module. A submodule N of M is called small y-closed (shortly, sy-closed) if M/N is s-nonsingular and denoted by $N \leq_{sy} M$.

Proposition (3.5): Let N be a submodule of an R-module M. Then the following statements are equivalent:

(i) scl (N) = N

(ii) N is sy-closed submodule of M.

Proof: (i) ⇒ (ii) Let$(0 \neq)\bar{x} \in Z^s\left(\frac{M}{N}\right)$, then there exists a s-essential right ideal I of R such that $\bar{x}I = 0$ and $\bar{x} = x + N$, where $x \in M$. So$(x+N)I = 0$, $xI + N = 0$ then $xI \subseteq N = scl(N)$. Therefore, $x \in scl(scl(N))$, then x∈N since N = scl(N). So $\bar{x} = 0$ which is a contradiction. Hence $Z^s\left(\frac{M}{N}\right) = 0$.

(ii) ⇒ (i) Let $x \in scl(N)$, then $[N:x] \leq_{se} R$ and $[N:x] = \{r \in R \mid xr \in N\} = \{r \in R \mid (x+N)r = N\}$. Hence $r_R(x+N) \leq_{se} R$ and therefore, $x + N \in Z^s\left(\frac{M}{N}\right) = 0$, then $x \in N$ so $scl(N) \subseteq N$. Then N is sy-closed submodule of M.

Now, by using the equivalent of sy-closed submodule of an R-module M, we can prove the following:

Theorem (3.6): Let M be an R-module and let N be a submodule of M, we have scl scl scl(N) = scl scl(N). In other words M/scl scl(N) is s-nonsingular.

Proof: Let $N \subseteq scl(N)$. Replacing N by scl(N) in part (i) of Lem.(3.2). We get sclscl(N)$\sim^{s}$(scl(N) + scl(0)) = scl(N), scl scl scl(N)$\sim^{s}$scl scl(N)$\sim^{s}$scl(N) applying part(ii), we obtain $sclsclscl(N) \subseteq sclscl(N)$, and hence $scl\,scl\,scl(N) = scl\,scl(N)$. □

Corollary (3.7): Let M be an R-module. Then $Z_2^s(M)$ is a sy-closed submodule in M.

Lemma (3.8): Every sy-closed submodule of an R-module M contain $Z_2^s(M)$.

Proof: Let N be sy-closed submodule of M and let $0 \subseteq N$ then $scl(0) \subseteq sclN = N$ then $Z_2^s(M) = scl\,scl(0) \subseteq scl\,scl\,N = scl\,N = N$. □

Remarks and Examples (3.9):

1. Every sy-closed submodule is s-closed.

Proof: let $A \leq M$ and $A \leq_{sy} M$, to show that $A \leq_{sc} M$. Suppose $A \leq_{se} B \leq M$ by Pro.(2.6), so $\frac{B}{A}$ is s-singular and by assumption $A \leq_{sy} M$, i.e. $\frac{M}{A}$ is s-nonsingular, and $\frac{B}{A} \leq \frac{M}{A}$, then $\frac{B}{A}$ is s-nonsingular and since $\frac{B}{A}$ is s-nonsingular and s-singular, so $\frac{B}{A} = 0$, $A = B$ then $A \leq_{sc} M$.

2. The converse of (1) is not be true, in general. For example: 0 is a s-closed submodule of any module M, but 0 is not sy-closed submodule of M.

3. If M is s-nonsingular, then every s-closed submodule is sy-closed.

Proof: Assume that M is a s-nonsingular R-module, and let A be an s-closed submodule in M. Put $Z^s\left(\frac{M}{A}\right) = \frac{B}{A}$, where B is a submodule of M, with $A \leq B$. Clearly $\frac{B}{A}$ is an s-singular module.

Now A ≤ B and M is a s-nonsingular module, therefore B is a s-nonsingular submodule of M. Then by Pro.(2.6), $A \leq_{se} B$. But A is an s-closed submodule in M, thus A = B, and $Z^s(\frac{M}{A}) = 0$, hence A is sy-closed submodule in M.

4. Every sy-closed submodule in M is y-closed.

Proof: suppose N is an sy-closed submodule in M. i.e. $Z^s(\frac{M}{N}) = 0$. Let $\bar{x} \in Z(\frac{M}{N})$, then there exists an essential right ideal I of R such that $\bar{x}I = 0$. And by [12], we get $\bar{x} \in Z^s\left(\frac{M}{N}\right) = 0$. Hence N is an y-closed submodule in M.

5. The converse of (4) is not true in general, for example: Consider Z_6 as Z_6-module, then $2Z_6 \leq_{\mathbf{y}} Z_6$ since $Z(\frac{Z6}{2Z6}) = 0$, but it is not sy-closed submodule in Z_6, since $Z^s(\frac{Z6}{2Z6}) = 3Z_6$.

6. If A ≤ B ≤ M, if $A \leq_{\mathbf{sy}} M$ then B need not be sy-closed submodule of M. For example: Consider Z as Z-module and $0 \leq 4Z \leq Z$. Clearly $0 \leq_{\mathbf{sy}} Z$ but $Z^s(\frac{Z}{4Z}) = Z^s(Z_4) = Z_4$ s-singular.

7. An epimorphic image of sy-closed submodule need not be sy-closed submodule as the following example show: let $\pi : Z \to \frac{Z}{2Z}$ be the natural epimorphism. Clearly $0 \leq_{sy} Z$, but π (0) = 0 is not sy-closed in $\frac{Z}{2Z}$ because $\frac{Z}{2Z} \cong Z_2$.

Proposition (3.10): Let M be an R-module and let A ≤ B ≤ M, then

1. If $A \leq_{\mathbf{sy}} M$, then $A \leq_{\mathbf{sy}} B$.

2. Let A ≤ B ≤ M, then $B \leq_{\mathbf{sy}} M$ if and only if $\frac{\mathbf{B}}{\mathbf{A}} \leq_{\mathbf{sy}} \frac{\mathbf{M}}{\mathbf{A}}$.

Proof: 1. Assume that $A \leq_{sy} M$, to show that $A \leq_{sy} B$, let b ∈ B such that $b+A \in Z^s\left(\frac{B}{A}\right)$. Therefore, b ∈ M then $b+A \in Z^s\left(\frac{M}{A}\right) = 0$. So b + A = A, then b ∈ A and hence $Z^s\left(\frac{B}{A}\right) = 0$.

2. Let m ∈ M if b ∈ B such that $(m+b) + A \in Z^s\left(\frac{M}{A}/\frac{B}{A}\right)$ by the third isomorphism theorem $(M / A)/(B / A) \cong M/B$, so $(m + b) + A \in Z^s(M/B) = 0$ so m + b ∈ A, then $Z^s\left(\frac{M/A}{B/A}\right) = 0$.

Proposition (3.11): Let A, B be a submodules of an R-module M, if $A \leq_{\mathbf{sy}} B$ and $B \leq_{\mathbf{sy}} M$, then $A \leq_{\mathbf{sy}} M$.

Proof: Let $A \leq_{\mathbf{sy}} B$ and $B \leq_{sy} M$. Now consider the following short exact sequence:

$0 \to \frac{B}{A} \xrightarrow{i} \frac{M}{A} \xrightarrow{\pi} \frac{M/A}{B/A} \to 0$. Where i is the inclusion map and π is the natural epimorphism. Since $A \leq B \leq_{sy} M$, then $\frac{B}{A} \leq_{sy} \frac{M}{A}$ by (Pro.(2.10)(2)), since $\frac{B}{A}$ and $\frac{M/A}{B/A}$ are s-nonsingular, then by module extension of s-nonsingular R-module is s-nonsingular, then $\frac{M}{A}$ is s-nonsingular. □

Proposition (3.12): Let f: M $\to$ N be an epimorphism and A $\leq_{sy}$ M. If ker f $\subseteq$ A, then f(A)$\leq_{sy}$ N.

Proof: Assume that A $\leq_{sy}$ M. To show that f (A) $\leq_{sy}$ N. Let n $\in$ N such that $r_R(n + f(A)) \leq_{se}$ R. Since f is an epimorphism, then n = f(m) , for some m$\in$ M . Since ker f $\subseteq$ A, then r_R (n + f(A)) $\subseteq$ r_R (m + A) and hence ann(n + f(A)) $\leq_{se}$ R, thus r_R (m + A) $\leq_{se}$ R but A $\leq_{sy}$ M, therefore m $\in$ A. Thus n = f(m) $\in$ f(A). □

Proposition (3.13): Let f: M $\to$ N be an R-homomorphim and B $\leq_{sy}$ N, then for every s-singular submodule A of M, f(A) $\subseteq$ B.

Proof: Let $\pi : N \to \frac{N}{B}$ be the natural epimorphisim. Consider $\pi \circ f: M \to \frac{N}{B}$. Now $\pi \circ f\,|_A : A \to \frac{N}{B}$ but A is s-singular and $\frac{N}{B}$ is s-nonsingular (since B $\leq_{sy}$ N) therefore $\pi \circ f\,|_A = 0$, thus $\pi(f(A)) = 0$ and hence f(A) $\subseteq$ ker π , f(A) $\subseteq$ B. □

Proposition (3.14): Let M be an R-module and A $\leq_{sy}$ M. Then Z^s (M) = Z^s (A).

Proof: It is enough to show that Z^s (M) $\subseteq$ Z^s (A). Let i: Z^s (M) $\to$ M be the inclusion map and $\pi : M \to \frac{M}{A}$ be the natural epimorphism. Consider the map $\pi \circ i : Z^s(M) \to \frac{M}{A}$. Since Z^s (M) is s-singular and $\frac{M}{A}$ is s-nonsingular (since A $\leq_{sy}$ M) then $\pi \circ i = 0$, (by Pro.(2.5). So $\pi\circ i\,(Z^s(M)) = \pi\left(Z^s(M)\right) = 0$. Thus Z^s (M) $\subseteq$ ker A = A. But $Z^s(A) = Z^s(M) \cap A$, therefore $Z^s(A) = Z^s(M)$. □

Proposition (3.15): Let M be an R-module and A $\leq_{sy}$ M. Then $\frac{\mathbf{M}}{\mathbf{B}}$ is s-singular if and only if B $\leq_{se}$ M.

Proof: Let A $\leq_{sy}$ M and $\frac{M}{B}$ is s-singular. By the third isomorphism theorem $\frac{M}{B} \cong \frac{M/A}{B/A}$ since $\frac{M}{B}$ is s-nonsingular by([12, Pro.(2.7)] $\frac{B}{A} \leq_{se} \frac{M}{A}$. Let $\pi : M \to \frac{M}{A}$ be the natural epimorphism B = $\pi^{-1}(\frac{B}{A}) \leq_{se} \pi^{-1}(\frac{M}{A})$ = M.

Conversely, let B $\leq_{sy}$ M and consider the following exact sequence $0 \to B \xrightarrow{i} M \to M/B \to 0$ and since i is s-essential monomorphism then by proposition M/A is s-singular. □

4. SY-Extending Modules

In this section, we introduce small-y-extending (shortly sy-extending), which is generalization of y-extending modules.

Definition (4.1): An R-module M called an sy-extending, if every sy-closed submodule is a direct summand.

Proposition (4.2): Every sy-closed submodule of sy-extending module is sy-extending.

Proof: Let M be sy-extending module and $A \leq_{sy} M$. We want to show that is sy-extending module. Let $K \leq_{sy} A$ and $A \leq_{sy} M$ then by Pro. (3.11) $K \leq_{sy} M$. But M is sy-extending, therefore K is a direct summand of M and by [10] K is a direct summand of A. □

Proposition (4.3): Any direct summand of sy-extending modules is a sy-extending module.

Proof: Suppose $M = K \oplus K'$ for some submodules K and K' of M. let L be a sy-closed submodule of K. Since $\frac{M}{L \oplus K'} = \frac{K \oplus K'}{L \oplus K'} \cong \frac{K}{L}$ then $L \oplus K'$ is a sy-closed submodule of M and M is sy-extending, so that $L \oplus K'$ is a direct summand of M which gives that L is a direct summand of M and since L a submodule of K . Then L is a direct summand of K. It follows that K is sy-extending module.

The following proposition gives a characterization of sy-extending modules.

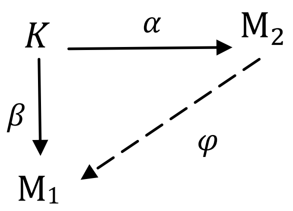
Proposition (4.4): An R-module M is sy-extending module if and only if every sy-closed submodule of M is s-essential in a direct summand.

Proof: (⇒) It is clear.

(⇐) Let $A \leq_{sy} M$, we want to show that A is a direct summand of M. Sine $A \leq_{sy} M$, then by our assumption $A \leq_{se} K$, where K is a direct summand of M. Thus K/A is s-singular by Pro. (2.6). But $K/A \subseteq M/A$ and M/A is s-nonsingular so K/A is s-nonsingular by Rem. (2.7) since K/A is s-singular and s-nonsingular. Then $A = K$ and hence A is a direct summand of M. Hence M is sy-extending module.

Theorem (4.5): Let $M = M_1 \oplus M_2$ be a direct sum of sy-extending modules M_1 and M_2 such that M_1 is M_2-injective. Then M is a sy-extending module.

Proof: Let N be a sy-closed submodule of M. Then M/N is s-nonsingular and $M_1/N \cap M_1 \cong M + N/N \subseteq M/N$. By Pro. (2.7) $M_1/N \cap M_1$ is s-nonsingular. Implies $N \cap M_1$ is sy-closed submodule of M_1 and M_1 is sy-extending so $N \cap M_1$ is a direct summand of M_1 and hence of M. It follows that $N \cap M_1$ is a direct summand of N so $N = (N \cap M_1) \oplus K$ for some submodule K of M. Let π_i: M→ M_i, i=1,2 denote the projection mapping. Consider the following diagram:



Where $\alpha = \pi_2|_K$ and $\beta = \pi_1|_K$. Note that α is a monomorphism and M_1 is M_2-injective. Thus, there exists a homomorphism $\varphi: M_2 \longrightarrow M_1$ such that φα = β. Let L={ x ∊ M_2: x+φ(x)} then it can easily be checked that L is a submodule of M and L ≅ M_2. Moreover, M = $M_1 \oplus L$. If k∊ K, then k $= m_1 + m_2$ for some $m_i \in M_i$, i=1,2. Then $m_1 = \beta(k) = \varphi\alpha(k) = \varphi(m_2)$, and this implies that $k = \varphi(m_2) + m_2 \in L$. Thus, K ⊆ L. Since $\frac{M}{N} = \frac{M_1}{N \cap M_1} \oplus \frac{L}{K}$, then L/K is s-nonsingular, so K is sy-closed submodule of L and L ≅ M_2 then K is a direct summand of L Thus, N is a direct summand of M, it follows that M is sy-extending module.

Recall that a submodule N of an R-module M is called fully invariant if f(N) ≤ N for each R-endomorphism f of M [7].

Proposition (4.7): Let M = $\oplus_{i \in I} M_i$ be an R-module, such that every sy-closed submodule of M is fully invariant, then M is sy-extending module if and only if M_i is sy-extending for each i∈I .

Proof: Clear that by Pro.(3.4). Conversely, let S be sy-closed submodule of M. For each i∊I, let $\pi_i : M \rightarrow M_i$ be the projection map. Now, let $x \in S$, then $x = \sum_{i \in I} m_i$, $m_i \in M_i$ and $m_i = 0$ for all but finite many element of i∈I. $\pi_i(x) = m_i$ for each i ∊ I. Since S is sy-closed, then by fully full invariance of S, $\pi_i(x) = m_i \in S \cap M_i$ so $x \in \oplus_{i \in I}(S \cap M_i)$. Thus $S \subseteq \oplus_{i \in I}(S \cap M_i)$. But $\oplus_{i \in I}(S \cap M_i) \subseteq S$, therefore S = $\oplus_{i \in I}(S \cap M_i)$. Since $S \leq_{sy}$ M, then by proposition (3.10) $S \cap M_i \leq_{sy} M_i$ for each i ∊ I, but M_i is sy-extending for each i ∊ I, therefore $(S \cap M_i)$ is a direct summand of M_i. Thus S is a direct summand of M.

□

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