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Oscillation of Solution of Third Order Non-Linear Neutral Difference Equations

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Abstract : this research shows the oscillatory of third -order difference equation $\Delta^2 z(n) + q(n) f(x(n+\mu)) = 0$.

Also ,deriving some new sufficient conditions which are of great value to the study of the target equation .

Keywords: Oscillation; neutral ; third order , difference equations;

Introduction :

In this article we shall concerned with the oscillatory of third –order non linear neutral delay difference equation as the from

$$\Delta^3 z(n) + q(n) f(x(n+\mu)) = 0 \qquad (1)$$

Where $z(n) = x(n) + p(n)x(n-\sigma)$,and n, σ, μ are positive integral ,the real sequences $\{p(n)\}, \{q(n)\}$ and f are satisfies following conditions

(d1) $q(n) \neq 0$, $p(n) \geq 0$

(d2) $f \in C(R, R)$ such that $f(xy) \geq f(x)f(y)$ and $\frac{f(y)}{y} \geq \gamma$ .where $\gamma > 0, y \neq 0.$

By a solution of (1) we mean a nontrivial real sequence $x(n)$ is said to be oscillatory ,if it neither eventually positive nor eventually negative , otherwise is nonoscillatory for $x(n)$ that is a defined for $n \geq n_0$

There has been a lot of attention in the oscillation of difference equation of first- order, second and higher order, see for examples [5-8,10-13] also we can find the results on third-order n [2-4,9,12,14,15] and the sources cited therein.

The our purpose in this paper is to supply sufficient condition which ensure equation (1) that all solution of this equation are oscillation .

In obtaining the chief results ,we will entail to use following lemma :

Lemma 1[3] :

Assume that g is positive real sequence and η is appositive integer .if

$$\liminf_{n\to\infty} \sum_{j=n}^{n+\eta-1} q(j) > \frac{\eta^{\eta}}{(\eta+1)^{\eta+1}}$$

Then

1) difference inequality

$$\Delta x(n) - q(n)x(n+\eta) \geq 0$$

has no eventually positive solution ,

2) difference inequality

$$\Delta x(n) - q(n)x(n+\eta) \leq 0$$

has no eventually negative solution.

Main Results :

In this portion ,we establish some new sufficient conditions for all solution of (1) to oscillation.

Theorem1 : assume that $0 \leq p_n < p < 1, \mu \geq 1 q(n) > 0$,,and

$$\limsup_{n\to\infty} \sum_{i=n_5}^{n} \left[\gamma h(i)q(i)f(1-p) - \frac{\left(\Delta h(i)\right)^2}{2(i+1)h(i)} \right] = \infty \quad (2)$$

$$\sum_{r=n}^{\infty}\sum_{j=n}^{\infty}\sum_{i=n}^{\infty} q(i) = \infty \qquad (3)$$

Then every solution of (1)oscillation

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Proof : Let $\{x(n)\}$ be an eventually positive solution of equation (1) then there exist an $n \geq n_0$, such that , $x(n-\sigma), x(n+\mu) > 0$.

Since $z(n) \geq x(n) > 0$ for $n_1 \geq n_0$

Form equation (1) we have

$$\Delta^3 z(n) = -q(n) f(x(n+\mu)) \leq 0$$

Then $\Delta^2 z(n)$ is monotone , and eventually of one sign.we calim that $\Delta^2 z(n) > 0$,suppose to the contrary that $\Delta^2 z(n) < 0$ then there exist anegative constant M for $n_2 \geq n_1$ so that $\Delta^2 z(n) < M$,$n \geq n_2$

We Summing from n_2 to $n-1$ we obtain

$$\Delta z(n) \leq \Delta z(n_2) + M(n-1-n_2) \qquad (4)$$

letting $n \to \infty$, then $\Delta z(n) \to -\infty$, there is an integer , $n_3 \geq n_2$ such that $\Delta z(n) \leq \Delta z(n_3) < 0$ for $n \geq n_3$.

Summing (4) from n_3 to $n-1$ we obtain

$$z(n) - z(n_3) \leq M(n-1-n_3)$$

this implies that $z(n) \to -\infty$ as $n \to \infty$,which a contradiction with actuality that $z(n)$ is positive .

Then $\Delta^2 z(n) > 0$,and we have two cases for $n \geq n_1$:

1) $\Delta^3 z_n \leq, \Delta^2 z_n > 0, \Delta z_n > 0, z_n > 0$,

2) $\Delta^3 z_n \leq, \Delta^2 z_n > 0, \Delta z_n < 0, z_n > 0$.

Case1 : by equation $z(n) = x(n) + p(n)x(n-\sigma)$ we get

$$x(n) \geq z(n) - pz(n-\sigma) \geq (1-p)z(n) \quad (5)$$

Using the above inequality in equation (1) as using (d2) we obtain

$$\Delta^3 z(n) + \gamma q(n) f(1-p) z(n+\mu) \leq 0 \qquad (6)$$

define $\omega(n) = \frac{h(n)\Delta^2 z(n)}{z(n+\mu)}$

$\omega(n)$ is positive and satisfies

$$\Delta\omega(n) = \frac{h(n)}{z(n+\mu)}\Delta^3 z(n) + \Delta\left(\frac{h(n)}{z(n+\mu)}\right)\Delta^2 z(n+1)$$

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Substation in (6) we have

$$\Delta\omega(n) \le -\gamma h(n)q(n)f(1-p) + \frac{\Delta h(n)}{h(n+1)}\omega(n+1) - \frac{h(n)\Delta z(n+\mu)\Delta^2 z(n+1)}{z(n+\mu)z(n+\mu+1)}$$

Since $z(n+\mu) \le z(n+\mu+1)$ then

$$\Delta\omega(n) \le -\gamma h(n)q(n)f(1-p) + \frac{\Delta h(n)}{h(n+1)}\omega(n+1) - \frac{h(n)\Delta z(n+\mu)\Delta^2 z(n+1)}{z^2(n+\mu+1)} \quad (7)$$

From the monotonicity property of $\Delta^2 z(n)$ we obtain $\Delta z(n) \ge \sum_{i=n_4}^{n-1} \Delta^2 z(n)$

Or $\Delta z(n) \ge \frac{n}{2}\Delta^2 z(n)$ for $n \ge 1 + n_4$

Then $\Delta z(n+1) \ge \frac{n+1}{2}\Delta^2 z(n+1)$

since $\Delta z(n)$ increasing then $\Delta z(n+\mu) \ge \Delta z(n+1)) \ge \frac{n+1}{2}\Delta^2 z(n+1)$ then the equation (7) it become

$$\Delta\omega(n) \le -\gamma h(n)q(n)f(1-p) + \frac{\Delta h(n)}{h(n+1)}\omega(n+1) - \frac{(n+1)h(n)\Delta^2 z(n+1)\Delta^2 z(n+1)}{2\, z^2(n+\mu+1)}$$

Then

$$\Delta\omega(n) \le -\gamma h(n)q(n)f(1-p) + \frac{\Delta h(n)}{h(n+1)}\omega(n+1) - \frac{(n+1)h(n)}{2h^2(n+1)}w^2(n+1)$$

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$$\Delta\omega(n) \leq -\left[\frac{\sqrt{(n+1)h(n)}}{\sqrt{2}\ h(n+1)}\omega(n+1) - \frac{\Delta h(n)}{\sqrt{2(n+1)h(n)}}\right]^2 - \gamma h(n)q(n)f(1-p) + \frac{\left(\Delta h(n)\right)^2}{2(n+1)h(n)}$$

The we have

$$\Delta\omega(n) \leq -\gamma h(n)q(n)f(1-p) + \frac{\left(\Delta h(n)\right)^2}{2(n+1)h(n)}$$

Summing from n_5 into n we get

$$\omega(n+1) - \omega(n_5) \leq -\sum_{i=n_5}^{n}\left[\gamma h(i)q(i)f(1-p) - \frac{\left(\Delta h(i)\right)^2}{2(i+1)h(i)}\right]$$

Which yelds

$$\sum_{i=n_5}^{n}\left[\gamma h(i)q(i)f(1-p) - \frac{\left(\Delta h(i)\right)^2}{2(i+1)h(i)}\right] \geq w(n_5)$$

Which is contradiction with (2).

Case 2: $\Delta^3 z(n) \leq, \Delta^2 z(n) > 0, \Delta z(n) < 0, z(n) > 0$

In this case $x(n)$ is eventually positive .as , $\Delta z(n) < 0$ eventually , $z(n)$ is positive and nonincreasing ,we have $z(n) \to \beta$ as $n \to \infty$ where $0 < \beta < \infty$,then there exist $\epsilon \in (0,1)$ such that $\beta - \epsilon \leq z(n) \leq z(n-\sigma) \leq \beta + \epsilon$.

So $x(n) \geq z(n) - px(n-\sigma) \geq z(n) - pz(n-\sigma) \geq \tau z(n)$

Where $\tau = [(\beta - \in) - p(\beta + \in)]/(\beta + \in)$

Using the above inequality in (1) we obtain

$$\Delta^3 z(n) + \gamma\tau q(n) z(n+\mu) \leq 0$$

And

$$\Delta^3 z(n) \leq -\gamma\tau q(n) z(n+\mu)$$

Summing the above inequality from n into ∞ we obtain

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$$-\Delta^2 z(n) \leq -\gamma\tau \sum_{i=n}^{\infty} q(i) z(i+\mu)$$

$$\Delta^2 z(n) \geq \gamma\tau \sum_{i=n}^{\infty} q(i) z(i+\mu)$$

Where $z(n+\mu) \geq \beta$ then

$$\Delta^2 z(n) \geq \gamma\tau\beta \sum_{i=n}^{\infty} q(i)$$

Since summing again from n to ∞ we get ,we have

$$-\Delta z(n) \geq \gamma\tau\beta \sum_{j=n}^{\infty} \sum_{i=n}^{\infty} q(i)$$

Again summing from n to ∞

$$z(n) \geq \gamma\tau\beta \sum_{r=n}^{\infty} \sum_{j=n}^{\infty} \sum_{i=n}^{\infty} q(i)$$

This is contradiction with (3)

Theorem 2: suppose that $0 < p(n) \leq p < 1, \mu \geq 1, q(n) \leq 0$,if

$$\sum_{\kappa=n-\mu}^{n} (n-\kappa-1)|q(r)| f(1-p) \geq \frac{2}{n} \tag{8}$$

$$\liminf_{n\to\infty} \sum_{n}^{n+\mu-1} \sum_{n}^{s-1} \gamma |q(n)| f(1-p) > \left(\frac{\mu}{\mu+1}\right)^{\mu+1} \tag{9}$$

Then all solution of (1) is oscillation .

Proof : Let $\{x(n)\}$ be an eventually positive solution of equation (1) then there exist an $n \geq n_0$, where $x(n-\sigma), x(n+\mu) > 0$.

Form equation (1) we get

$$\Delta^3 z(n) = |q(n)| f(x(n+\mu)) \geq 0 \tag{10}$$

Since $(n) \geq x(n) > 0$, for $n_1 \geq n_0$ and by (10) we have two cases:

1) $\Delta^3 z(n) \geq 0, \Delta^2 z(n) > 0, \Delta z(n) > 0, z(n) > 0,$

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2) $\Delta^3 z(n) \geq 0, \Delta^2 z(n) < 0, \Delta z(n) > 0, z(n) > 0,$

Case 1: $\Delta^3 z(n) \geq 0, \Delta^2 z(n) > 0, \Delta z(n) > 0, z(n) > 0$, for $n_3 \geq n_2$

from (corollary 1.86 in [1]) we have

$$\Delta^\eta z(n) = \sum_{i=m}^{\xi-1} \frac{(n-a)}{(i-\eta)!} \Delta^i z(a) + \frac{1}{(\xi-\eta-1)!} \sum_{\kappa=a}^{n-\xi+\eta} (n-\kappa-1)^{(\xi-\eta-1)} \Delta^\xi z(\kappa)$$

Where $0 \leq \eta \leq \xi - 1$Take $\xi = 3, \eta = 1$ we obtain

$$\Delta z(n) = \sum_{i=1}^{2} \frac{(n-a)}{(i-\eta)!} \Delta^i z(a) + \sum_{r=a}^{n-2} (n-\kappa-1)\, \Delta^3 z(\kappa)$$

$$\Delta z(n) \geq \sum_{\kappa=a}^{n-2} (n-\kappa-1)\, \Delta^3 z(\kappa) \qquad (11)$$

Now from (5) and (10)we get

$$\Delta^3 z(n) \geq |q(n)| f\big((1-p) z(n+\mu)\big) \geq |q(n)| f(1-p) z(n+\mu)$$

Substation in (11) we get

$$\Delta z(n) \geq \sum_{\kappa=a}^{n-2} (n-\kappa-1) |q(\kappa)| f(1-p) z(\kappa+\mu)$$

And

$$\Delta z(n) \geq \sum_{\kappa=n-\mu}^{n} (n-\kappa-1) |q(\kappa)| f(1-p) z(\kappa+\mu)$$

So $z(n)$ increasing then

$$\Delta z(n) \geq z(n) \sum_{\kappa=n-\mu}^{n} (n-\kappa-1) |q(\kappa)| f(1-p)$$

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No by using $z(n) \geq \frac{n}{2}\Delta z(n)$ then

$$z(n) \geq \frac{n}{2}\Delta z(n) \geq \frac{n}{2}z(n)\sum_{\kappa=n-\mu}^{n}(n-\kappa-1)|q(\kappa)|f(1-p)$$

$$\frac{2}{n} \geq \sum_{\kappa=n-\mu}^{n}(n-\kappa-1)|q(\kappa)|f(1-p)$$

Which is contradiction with (8)

<u>Case 2:</u> $\Delta^3 z(n) \geq 0, \Delta^2 z(n) < 0, \Delta z(n) > 0, z(n) > 0$, for $n_4 \geq n_3$

Since equation (1) we can written as the from

$$\Delta^2 z(n+1) - \Delta^2 z(n) = |q(n)|f\big(x(n+\mu)\big) \quad (12)$$

since $x(n) \geq (1-p)z(n)$ and using (d2) then (10) it become

$$-\Delta^2 z(n) \geq \gamma|q(n)|f(1-p)z(n+\mu) \quad (13)$$

Summing both side of (13) from n to $s-1$ we get

$$-\Delta z(s) + \Delta z(n) \geq \textstyle\sum_{n}^{s-1}\gamma|q(i)|f(1-p)z(i+\mu)$$

And $$\Delta z(n) \geq \textstyle\sum_{n}^{s-1}\gamma|q(i)|f(1-p)z(i+\mu)$$

$$\Delta z(n) \geq z(n+\mu)\sum_{n}^{s-1}\gamma|q(i)|f(1-p)$$

$$\Delta z(n) - z(n+\mu) \geq \sum_{n}^{s-1}\gamma|q(i)|f(1-p)$$

Which is a contradiction of lemma (1) and condition (9).

Examples :

In this part ,we present some examples to illustrate the essential results

Example 1: consider third order delay difference equation

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$$\Delta^3\left[x(n)+\frac{1}{4}x(n-1)\right]+\left[\frac{6}{x(n+2)}\right]=0 \quad ,n>1$$

$$p(n)=\frac{1}{4}\ ,q(n)=6\,,f\big(x(n+2)\big)=\left[\frac{1}{x(n+2)}\right],\gamma=1,\mu=2,\sigma=1$$

We can find that all the condition of Th (1) are satisfies also we can find a oscillation solution by $(-1)^n$ from this equation.

Example 2: consider third order nutral difference equation

$$\Delta^3\left[x(n)+\frac{1}{2}x(n-2)\right]-12\big(x(n+3)\big)^3=0 \quad ,n>2$$

$$p(n)=\frac{1}{2}\ ,q(n)=-12\,,f\big(x(n+3)\big)=\big(x(n+3)\big)^3,\gamma=1$$

We can find that all the condition of Th (2) are satisfies also we can find a oscillation solution by $(-1)^n$ from this equation.

Conclusion: therein this paper, several necessary and sufficient conditions ,it is obtained to ensure the oscillation behavior of all solution of third order neutral difference equation .

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