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*P- Reslovable Spaces and Precompatible ideals

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Abstract

The intend of this paper is to introduce *P-resolvable spaces and precompatible ideals. Relation between *P-resolvable,I-resolvable and resolvable spaces are analysed. Many characterizations and properties of these two concepts are also obtained.

*Keywords :resolvable space, I -resolvable space, *P-resolvable space, *P-dense, I- Hausdorff.*

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1 Introduction and Preliminaries

Th notion of resolvable space was introduced by Hewitt. Many mathematicians such as Ganster[16], P.Sharma , S.Sharma[45], Comfort and Feng[7] are working in this area and found many interesting properties about resolvable spaces. Recently Comfort and van Mill[9] and also Comfort, Masavea and Zhou[10] studied resolvability of topological groups. The concept of compatible ideals was introduced by O.Njastad in 1966.This ideal was also called as supercompact by R.Vaidyanathaswamy. Further D.Jankovic and T.R. Hamlett are also worked in this area.

Throughout this paper, cl(A) and int(A) denote the closure of A and the interior of A respectively. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies

(i) A $\in$ I and B $\subseteq$ A implies B $\in$ I.

(ii) A $\in$ I, B $\in$ I implies A $\cup$ B $\in$ I.

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If $\{A_i, i = 1, 2, 3, ...\} \subseteq I$, then $\cup\{A_i : i = 1, 2, 3, ...\} \in I$(countable additivity) a topological ideal which satisfies also this condition is called σ-ideal. An ideal topological space denoted by (X, τ, I) is a topological space (X, τ) with an ideal I on X. For a subset A of X, $A^{*P}(I) = \{x \in X : U \cap A \notin I$ for each pre neighbourhood U of $x\}$ is called the pre local function of A with respect to I and τ. Some important ideals in a topological space are 1) the ideal of all finite subsets F

2) the ideal of all countable subsets C

3) the ideal of all nowhere dense sets N

4) the ideal of all meager sets M

5) the ideal of all Lebesgue null sets L

For every ideal topological space (X, τ, I), there exists a topology $\tau^{*P}(I)$,finer than τ, generated by the base $B(PO(X), I) = \{U\backslash G : U \in PO(X)$ and $G \in I\}$. Additionally, $\mathrm{cl}^{*P}(A) = A \cup A^{*P}$ defines a closure operator for $\tau^{*P}(I)$.

Definition 1.1 *A non-empty topological space (X,τ) is called resolvable if X is the disjoint union of two dense subsets.*

Definition 1.2 *A non-empty subset of an ideal topological space, $A \subseteq X$ is said to *P-dense, if $A^{*P}= X$.*

Definition 1.3 *A non-empty subset of an ideal topological space, $A \subseteq X$ is said to τ^{*P}-dense, if $cl^{*P}(A)= X$.*

2 *P- resolvable spaces

Definition 2.1 *A non-empty ideal topological space (X,τ,I) is called *P-resolvable if X is the union of two I^{*P}-dense subsets, where A and B are disjoint modulo an ideal(i.e) their intersection is in I.*

Proposition 2.2 *For a non empty ideal topological space (X,τ,I) the following conditions are equivalent.*

1) (X,τ) is resolvable space.

*2) (X,τ, I) where I =$\emptyset$ ideal is *P-resolvable*

Proof. *Straightforward.*

Proposition 2.3 *Every I^{*P}-dense set is τ^{*P}-dense set.*

Proof. *Let $A \subseteq X$ be I^{*P}-dense, then $A^{*P} = X$ which implies $cl^{*P}(A) = A \cup A^{*P} = X$. Thus A is τ^{*P}-dense set.*

Definition 2.4 *An ideal topological space(X,τ,I) is said to be *P-irresolvable, if it is not *P-resolvable.*

Remark 2.5 *An ideal topological space is *P-irresolvable if,*

1)I is maximum ideal(i.e)I = P(X)

2)I contains any non empty preopen set.

Theorem 2.6 *For a non empty ideal topological space (X,τ,I) X=X^{*P} if and only if the given ideal is completely codense ideal.*

Proof. *Straightforward.*

Theorem 2.7 *If (X,τ,I) is *P-resolvable, then I is completely codense.*

Proof. *If $X = A \cup B$ where A and B are I^{*P}-dense set, then $A^{*P}(I) = X$ and $B^{*P}(I) = X$. Therefore PO(X) $\cap$ A $\notin$ I and PO(X) $\cap$ B $\notin$ I. Hence PO(X)$\cap$ I = $\emptyset$.(i.e) I is completely codense.*

Corollary 2.8 *If an ideal topological space is (X, τ,I) is *P-resolvable, then I is codense.*

Proof. *Given (X, τ, I) be a *P-resolvable space by Theorem 6.1.7 then I is completely codense, we know that every completely codense is codense. Hence the result.*

Theorem 2.9 *Every *P-resolvable space is I-resolvable space.*

Proof. *Let $X = A \cup B$ where A and B are disjoint modulo an ideal. Since the given space is *P-resolvable then $A^{*P} = X$ and $B^{*P} = X$ (i.e) $A^{*P} \subseteq A*$ and $B^{*P} \subseteq B*$, then by the hypothesis, we get A^{*} and $B^{*} = X$. Hence $X = A \cup B$ where A and B are I-dense subsets. Therefore X is I-resolvable.*

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Theorem 2.10 *If an ideal topological space (X,τ,I) is *P-resolvable, then (X,τ,I) is I- Hausdorff.*

Proof. *Let A and B be disjoint I^{*P}-dense subsets of X such that $X = A \cup B$. Since $A^{*P} = X$ and $B^{*P} = X$ implies A^* and $B^* = X$. Therefore $A \subseteq int(A^*)$, similarly B, then A and B are I-open sets. Let x,y be any two element of X and both x and y are in A. Take $U = A \setminus \{y\}$ and $V = B \cup \{y\}$ then U and V are also disjoint I-open sets containing x and y in U and V respectively.*

Definition 2.11 *An ideal topological space is said to be *P- Hausdorff , if for every point x,y $\in$ X, there exists two disjoint I^{*P}-open sets containing each respectively.*

Theorem 2.12 *If (X,τ,I) is *P-resolvable, then (X,τ,I) is I^{*P}- Hausdorff.*

Proof. *Let A and B be disjoint I^{*P}-dense subsets of X such that $X = A \cup B$. Since $A^{*P} = X$ and $B^{*P} = X$. Therefore $A \subseteq int(A^{*P})$, similarly B, then A and B are I^{*P}-open sets. Let x,y be any two element of X and both x and y are in A. Take $U = A \setminus \{y\}$ and $V = B \cup \{y\}$ then U and V are also disjoint I^{*P}-open sets containing x and y in U and V respectively.*

Remark 2.13 *Let (X,τ,I) be an ideal topological space then,*
*1) If τ and τ_1 are any topologies with $\tau \subseteq \tau_1$, if (X, τ, I) is *P-resolvable then (X, τ_1, I) is *P-resolvable space.*
*2) If τ and τ_1 are any topologies with $\tau \subseteq \tau_1$, if (X, τ, I) is *P-resolvable then (X, τ_1, I) is I-resolvable space.*

3 Precompatibility

Definition 3.1 *Let (X,τ) be a topological space with I an ideal on X, then τ is pre-compatible with I if for every $A \subseteq X$ and for every $x \in A$ there exists a $U \in PO(x)$ such that $U \cap A \in I$, then $A \in I$ and it is denoted by $\tau \sim_P I$*

Theorem 3.2 *If (X, τ,I) is an ideal topological space, then the following statements are equivalent*

1. $\tau \sim_P I$
2. *If* A *has a cover of preopen sets each of whose intersection with* A *is in* I*, then* A *is in* I
3. *For every,* $A \subseteq X$*,* $A \cap A^{*P} = \emptyset$ *implies* $A \in I$
4. *For every,* $A \subseteq X$*,* $A - A^{*P} \in I$
5. *For every* τ^{*P}*-closed subset* A*,* $A - A^{*P} \in I$
6. *For every* $A \subseteq X$*, if* A *contains no nonempty subset* B *with* $B \subseteq B^{*P}$*, then* $A \in I$

Proof 1) $\Rightarrow 2) \Rightarrow 3)$ *are straightforward from the definitions.*

3) $\Rightarrow$ *4) Let* $A \subseteq X$*, then by 3)* $(A - A^{*P}) \cap (A - A^{*P})^{*P} = \emptyset$ *which implies* $A - A^{*P} \in I$

4) $\Rightarrow$ *5) Since 4) is true for every* $A \subseteq X$ *it is also true for every* τ^{*P}*-closed subset* A *such that* $A - A^{*P} \in I$

5) $\Rightarrow$ *1) For* $A \subseteq X$ *and for every* $x \in A$ *there exists* $U \in PO(x)$ *such that* $U \cap A \in I$*,then* $A \cap A^{*P} = \emptyset$
To prove $A \in I$*: Since* $A \cap A^{*P} = \emptyset \Rightarrow A^{*P} = \emptyset$*. Then* $(A \cup A^{*P})$ *is* τ^{*P}*-closed by 5)* *we get* $(A \cup A^{*P}) - ((A \cup A^{*P})^{*P}) \in I$
$(A \cup A^{*P}) - ((A \cup A^{*P})^{*P}) \in I = (A \cup A^{*P}) - A^{*P} \cup (A^{*P})^{*P}$
$= (A \cup A^{*P}) - A^{*P} = A$ *Therefore* $A \in I$

4) $\Rightarrow$ *6)* $A \subseteq X$ *and* A *has no nonempty subset* B *with* $B \subseteq B^{*P}$*. From 4) we get* $A - A^{*P} \in I$*,* $A \cap A^{*P} \subseteq (A \cap A^{*P})^{*P}$*. Since* $(A \cap A^{*P})^{*P} = A^{*P}$ *and hence* $A \cap A^{*P} = \emptyset$*. Therefore* $A = A - A^{*P}$ *and* $A \in I$

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6) $\Rightarrow$ 4) For $A \subset X$, Since $(A - A^{*P}) \cap (A - A^{*P})^{*P} = \emptyset$ then $A - A^{*P}$ contains no non-empty subset B with $B \subseteq B^{*P}$. Therefore by 6) we have $A - A^{*P} \in I$.

Theorem 3.3 *Let (X,τ) be a space with I an ideal on X. Then the following are equivalent if $\tau \sim_P I$, then*

1. *For every $A \subseteq X$, $A \cap A^{*P} = \emptyset \Rightarrow A^{*P} = \emptyset$*
2. *For every $A \subseteq X$,$(A - A^{*P})^{*P} = \emptyset$*
3. *For every $A \subseteq X$, $(A \cap A^{*P})^{*P} \subseteq A^{*P}$*

Proof: (1) and (2) follow from (3) and (4) of Theorem 6.1.15.
3) by 1) $(A \cap A^{*P})^{*P} \subseteq A^{*P} \cap (A^{*P})^{*P}$
$(A \cap A^{*P})^{*P} \subseteq A^{*P}$

Theorem 3.4 *Let (X,τ) be a space and I an ideal on X such that $\tau \sim_P I$. A set is closed with respect to τ^{*P} if and only if it is the union of a set which is preclosed with respect to τ and a set in I.*

Proof. Let A be τ^{*P}-closed then $cl^{*P}(A) = A$ since $cl^{*P}(A) = A \cup A^{*P} = A$, (i.e) $A^{*P} \subseteq A$
$A = (A - A^{*P}) \cup A^{*P}$ where $A - A^{*P} \in I$ by Theorem 6.1.15(4) and A^{*P} is preclosed.

Conversely, if $A = B \cup I_1$, where B is preclosed and $I_1 \in I$ then $A^{*P} = B^{*P} \cup I_1^{*P}$, always $I_1^{*P} = \emptyset$. $\Rightarrow A^{*P} = B^{*P} \subseteq pcl(B) = B \subseteq A$, thus $A^{*P} \subseteq A$ and therefore A is τ^{*P}-closed.

Theorem 3.5 *Let (X,τ) be a space with an ideal I on X such that $\tau \sim_P I$ and $\{x\} \in I$ for each $x \in X$. If a set $A \subseteq X$ is closed with respect to τ^{*P}, then A is the union of a set which is *P-perfect with respect to τ and a set in I.*

Proof. Let $A \subseteq X$ be closed with respect to τ^{*P} then by Theorem6.1.17 A can be written as a union of $A = A^{*P} \cup I_1$ with $I_1 \in I$
$A^{*P} - (A^{*P})^{*P} \subseteq (A - A^{*P})^{*P}$ we have $(A - A^{*P})^{*P} = \emptyset$ by theorem 6.1.16(2). Therefore $A^{*P} - (A^{*P})^{*P} = \emptyset$ implies $A^{*P} = (A^{*P})^{*P}$.

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Definition 3.6 *If I and J are ideals on (X,τ), then the pre-extension of I via J denoted by* $I\ {}^*p\ J = \{A \subseteq X : A^{*P}(I) \in J\}$

Definition 3.7 *The pre-extension of I over the ideal of nowhere dense sets N is denoted by* $I_p^{\sim} = \{A \subseteq X,\ int(A^{*P}(I) = \emptyset\}$

Definition 3.8 *The countable extension of I denoted by* $I_{\sigma P}^{\sim} = \{A \subseteq X : A = \cup_{n\in N} A_n,\ A_n \in I_p^{\sim}$ *for each* $n \in N\}$

Remark 3.9 *Every extension of I via J(I * J) is pre-extension of I via J (I *p J).*

Proof. $I * J = \{A \subseteq X : A^*(I) \in J\}$. *Always* $A^{*P}(I) \subseteq A^*(I)$ *and also J is an ideal. Therefore, if* $A^*(I) \in J$, *then all subsets of* $A^*(I)$ *are also element of J.(i.e)* $A^{*P}(I) \in J$. *Therefore* $I * J$ *implies* $I * pJ$.

Theorem 3.10 *Let* (X, τ, I) *be an ideal topological space, if* $\beta = I \cup N$ *then for every subset A of X,*

1. $(A^{*P}(I))^{*P}(N) = (A^{*P}(I * pN))^{*P}(N) = (A^{*P}(I_p^{\sim}))^{*P}(N) = A^{*P}(I_p^{\sim})$
2. $(A^{*P}(I))^{*P}(N) = A^{*P}(\beta))^{*P}(N)$
3. $(A^{*P}(\beta))^{*P}(N) \subseteq A^{*P}(N)$

Proof: (1) First we prove $(A^{*P}(I))^{*P}(N) \subset A^{*P}(I_p^{\sim})$. *Let* $x \notin A^{*P}(I_p^{\sim})$. *Then there exists an preopen set G containing x such that* $G \cap A \in I_p^{\sim}$ *and so* $(G\cap A)^{*P}(I) \in N \Rightarrow G \cap A^{*P}(I) \in N$, *which implies* $x \notin (A^{*P}(I))^{*P}(N)$
$(A^{*P}(I))^{*P}(N) \subset A^{*P}(I_p^{\sim}) \subset A^{*P}(I)$
then $((A^{*P}(I))^{*P}(N))^{*P}(N) \subset (A^{*P}(I_p^{\sim}))^{*P}(N) \subseteq (A^{*P}(I))^{*P}(N)$
since $N{\sim_p}\ \tau$ *(i.e) nowhere dense ideal is compatible* $\Rightarrow (A^{*P}(I_p^{\sim}))^{*P}(N) = (A^{*P}(I))^{*P}(N)$
Thus $(A^{*P}(I_p^{\sim}))^{*P}(N) = A^{*P}(I{\sim_p})$ *Hence proved.*

2) Since $I \subset \beta \subset I_p^{\sim}$

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$A^{*P}(I_p^{\sim}) \subset A^{*P}(\beta) \subset A^{*P}(I)$
$(A^{*P}(I_p^{\sim}))^{*P}(N) \subseteq A^{*P}(\beta))^{*P}(N) \subseteq (A^{*P}(I))^{*P}(N)$
Therefore by (1), they are equal.

3) Since $N \subset \beta$
implies $A^{*P}(\beta) \subset (A^{*P})(N)$
$((A^{*P}(\beta))^{*P}(N) \subset ((A^{*P})(N))^{*P}(N) = A^{*P}(N)$
Therefore, $((A^{*P}(\beta))^{*P}(N) = A^{*P}(N)$.

Theorem 3.11 *An ideal space (X,*τ*,I) is *P-resolvable if and only if it is* $I_p^{\sim}$*-resolvable.*

*Proof: Suppose X is *P-resolvable. Let A be any* I^{*P}*-dense subset of X, then* $A^{*P}(I) = X$ *and so* $(A^{*P}(I))^{*P}(N) = X^{*P}(N)$
*[X is *P-resolvable implies I is completely codense then X*$\subseteq X^{*P}$*]*
Therefore $X^{*P}(N) = X$ *by Theorem 6.1.23* $(A^{*P}(I))^{*P}(N) = A^{*P}(I_p^{\sim})$
Hence $A^{*P}(I_p^{\sim}) = X$ *which implies X is* $I_p^{\sim}$*-resolvable.*

Converse part is proved using the relation between the ideals.(i.e) $I \subset I * pN$ *implies* $I \subset I_p^{\sim}$*. Therefore* $A^{*P}(I_p^{\sim}) \subseteq A^{*P}(I)$*. Since the space is* $I_p^{\sim}$*-resolvable* $A^{*P}(I_p^{\sim}) = X$ *implies* $A^{*P}(I) = X$ *then the space is also *p-resolvable space.*

Theorem 3.12 *If* (X, τ, I) *is an ideal space, I is precompatible and* $\beta = I \cup N$*, then* $\beta = \beta * pN$

Proof: Let $A \in \beta * pN \Rightarrow A^{*P}(\beta) \in N$ *which implies* $((A^{*P})(\beta))^{*P}(N) = \emptyset$
Using theorem 6.1.23 (2) we have $(A^{*P}(I))^{*P}(N) = \emptyset$ *then* $A^{*P}(I) \in N$ *and so* $A \cap A^{*P}(I) \in N$*. Since I is precompatible,* $A - A^{*P}(I) \in I$*. Hence* $A = (A - A^{*P}(I)) \cup (A \cap A^{*P}(I))$ *which implies* $A \in I \cup N = \beta$*. Therefore* $\beta * pN \subseteq \beta$ *and always* $\beta \subseteq \beta * pN$*. Therefore* $\beta = \beta * pN$*.*

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