Bounds for the Zeros of a Certain Class of Polynomials

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Abstract: In this paper we find bounds for the zeros of a certain class of polynomials whose coefficients or their real and imaginary parts are restricted to certain conditions. Our results improve and generalize many known results in this direction.

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1. Introduction and Statement of Results

The following result, known as the Enestrom-Kakeya Theorem [4,5] , is well-known in the theory of distribution of zeros of polynomials:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \leq 1$.

In the literature, there exist several generalizations and extensions of this result. Joyal, Labelle and Rahman [3] extended it to polynomials with general monotonic coefficients by proving the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then P(z) has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

Aziz and Zargar [1] relaxed the hypothesis of Theorem A and proved

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$ka_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then P(z) has all its zeros in the disk

$$|z+k-1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

On the other hand, Y. Choo [2] proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda \neq 1$, $1 \le k \le n$, $a_{n-k} \neq 0$,

$$a_n \ge a_{n-1} \ge ...... \ge a_{n-k+1} \ge \lambda a_{n-k} \ge a_{n-k-1} \ge ...... \ge a_1 \ge a_0 .$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of P(z) lie in the disk $|z| \le k_1$, where k_1 is the positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0,$$

with

$$\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n} \quad \text{and} \quad \delta_1 = \frac{a_n + (\lambda-1)a_{n-k} - a_0 + |a_0|}{|a_n|} .$$

If $a_{n-k} > a_{n-k+1}$, then all the zeros of P(z) lie in the disk $|z| \le k_2$, where k_2 is the positive root of the equation

$$K^k - \delta_2 K^k - |\gamma_2| = 0,$$

with

$$\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n} \quad \text{and} \quad \delta_2 = \frac{a_n + (1-\lambda)a_{n-k} - a_0 + |a_0|}{|a_n|} .$$

In this paper, we prove some more general results, which include many generalizations and extensions of Enestrom-Kakeya Theorem as special cases. We first prove

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with such that for some $\rho, \sigma \neq 1$,$0 < \tau \le 1$, $1 \le k \le n$, $a_{n-k} \neq 0$,

$$\rho + a_n \ge a_{n-1} \ge ...... \ge a_{n-k+1} \ge \sigma a_{n-k} \ge a_{n-k-1} \ge ...... \ge a_1 \ge \tau a_0 .$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of P(z) lie in the disk $|z| \le k_3$, where k_3 is the positive root of the equation

$$K^{k+1} - \delta_3 K^k - |\gamma_3| = 0,$$

with

$$\gamma_3 = \frac{(\sigma-1)a_{n-k}}{a_n} \quad \text{and} \quad \delta_3 = \frac{|\rho| + \rho + a_n + (\sigma-1)a_{n-k} + 2|a_0| - \tau(|a_0| + a_0)}{|a_n|} .$$

If $a_{n-k} > a_{n-k+1}$, then all the zeros of P(z) lie in the disk $|z| \le k_4$, where k_4 is the positive root of the equation

$$K^k - \delta_4 K^k - |\gamma_4| = 0,$$

with

$$\gamma_4 = \frac{(1-\sigma)a_{n-k}}{a_n}$$

and

$$\delta_4 = \frac{|\rho| + \rho + a_n + (1-\sigma)a_{n-k} + 2|a_0| - \tau(|a_0| + a_0)}{|a_n|} .$$

Remark 1: If we take ρ =0 and $\tau = 1$ in Theorem 1, we get Theorem D. Many other interesting results can be obtained from Theorem 1 by taking different values of the parameters ρ, σ, τ and k. For instance, if ρ =0, we have the following result

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some, $\sigma \neq 1$,$0 < \tau \leq 1$, $1 \leq k \leq n$, $a_{n-k} \neq 0$,

$$a_n \geq a_{n-1} \geq ...... \geq a_{n-k+1} \geq \sigma a_{n-k} \geq a_{n-k-1} \geq ...... \geq a_1 \geq \tau a_0 .$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of P(z) lie in the disk $|z| \leq k_5$, where k_5 is the positive root of the equation

$$K^{k+1} - \delta_5 K^k - |\gamma_5| = 0,$$

with

$$\gamma_5 = \frac{(\sigma - 1)a_{n-k}}{a_n}$$

and

$$\delta_5 = \frac{a_n + (\sigma - 1)a_{n-k} + 2|a_0| - \tau(|a_0| + a_0)}{|a_n|} .$$

If $a_{n-k} > a_{n-k+1}$, then all the zeros of P(z) lie in the disk $|z| \leq k_6$, where k_6 is the positive root of the equation

$$K^k - \delta_6 K^k - |\gamma_6| = 0,$$

with

$$\gamma_6 = \frac{(1 - \sigma)a_{n-k}}{a_n}$$

and

$$\delta_6 = \frac{a_n + (1 - \sigma)a_{n-k} + 2|a_0| - \tau(|a_0| + a_0)}{|a_n|} .$$

If the coefficients of the polynomial P(z) are complex, then we have the following result of independent interest:

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j, \text{Im}(z) = \beta_j, j = 0,1,......,n$ such that for some $\rho, \sigma \neq 1$,$0 < \tau \leq 1$, $1 \leq k \leq n$, $\alpha_{n-k} \neq 0$,

$$\rho + \alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_{n-k+1} \geq \sigma\alpha_{n-k} \geq \alpha_{n-k-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0 .$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then all the zeros of P(z) lie in the disk $|z| \leq k_7$, where k_7 is the positive root of the equation

$$K^{k+1} - \delta_7 K^k - |\gamma_7| = 0,$$

with

$$\gamma_7 = \frac{(\sigma-1)\alpha_{n-k}}{a_n}$$

and

$$\delta_7 = \frac{|\rho| + \rho + \alpha_n + (\sigma-1)\alpha_{n-k} + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1}|\beta_j|}{a_n} .$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then all the zeros of P(z) lie in the disk $|z| \le k_8$, where k_8 is the positive root of the equation

$$K^k - \delta_8 K^k - |\gamma_8| = 0,$$

with

$$\gamma_8 = \frac{(1-\sigma)\alpha_{n-k}}{a_n}$$

and

$$\delta_8 = \frac{|\rho| + \rho + \alpha_n + (1-\sigma)\alpha_{n-k} + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1}|\beta_j|}{a_n} .$$

Remark 2: If the coefficients a_j in Theorem 2 are real i.e. $\beta_j = 0$ for all j, then it reduces to Theorem 1.

For different values of the parameters ρ, σ, τ and k, we get many other interesting results. For example, if we take ρ =0, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j, \text{Im}(z) = \beta_j, j = 0,1,......,n$ such that for some $\sigma \ne 1$,$0 < \tau \le 1$, $1 \le k \le n$, $\alpha_{n-k} \ne 0$,

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{n-k+1} \ge \sigma\alpha_{n-k} \ge \alpha_{n-k-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0 .$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then all the zeros of P(z) lie in the disk $|z| \le k_9$, where k_9 is the positive root of the equation

$$K^{k+1} - \delta_9 K^k - |\gamma_9| = 0,$$

with

$$\gamma_9 = \frac{(\sigma-1)\alpha_{n-k}}{a_n}$$

and

$$\delta_9 = \frac{\alpha_n + (\sigma-1)\alpha_{n-k} + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1}|\beta_j|}{a_n} .$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then all the zeros of P(z) lie in the disk $|z| \le k_{10}$, where k_{10} is the positive root of the equation

$$K^k - \delta_{10} K^k - |\gamma_{10}| = 0,$$

with

$$\gamma_{10} = \frac{(1-\sigma)\alpha_{n-k}}{a_n}$$

and

$$\delta_{10} = \frac{\alpha_n + (1-\sigma)\alpha_{n-k} + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1}|\beta_j|}{a_n} \quad .$$

2. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... \\ &\quad + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \end{aligned}$$

If $a_{n-k-1} > a_{n-k}$, then $a_{n-k+1} > a_{n-k}$ and we have

$$\begin{aligned} F(z) = -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... \\ + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (\sigma a_{n-k} - a_{n-k-1})z^{n-k} - (\sigma - 1)a_{n-k} z^{n-k} \\ + ... + (a_2 - a_1)z^2 + (a_1 - \tau a_0)z + (\tau - 1)a_0 + a_0 \end{aligned}$$

Therefore, for $|z| > 1$,

$$\begin{aligned} |F(z)| &\geq |a_n z^{n+1} + (\sigma - 1)a_{n-k} z^{n-k}| - |-\rho z^n + (\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... \\ &\quad + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (\sigma a_{n-k} - a_{n-k-1})z^{n-k} - (\sigma - 1)a_{n-k} z^{n-k} + ... \\ &\quad + (a_2 - a_1)z^2 + (a_1 - \tau a_0)z + (\tau - 1)a_0 + a_0 \ | \\ &\geq |z|^{n-k}|a_n z^{k+1} + (\sigma - 1)a_{n-k}| - |z|^n[|\rho| + (\rho + a_n - a_{n-1}) + \frac{a_{n-1} - a_{n-2}}{|z|} + ... \\ &\quad + \frac{a_{n-k+1} - a_{n-k}}{|z|^{k-1}} + \frac{\sigma a_{n-k} - a_{n-k-1}}{|z|^k} + ... + \frac{|a_2 - a_1|}{|z|^{n-2}} \\ &\quad + \frac{a_1 - \tau a_0}{|z|^{n-1}} + \frac{(1-\tau)|a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}] \\ &> |z|^{n-k}|a_n z^{k+1} + (\sigma - 1)a_{n-k}| - |z|^n[|\rho| + (\rho + a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + ... \\ &\quad + (a_{n-k+1} - a_{n-k}) + (\sigma a_{n-k} - a_{n-k-1}) + ... + (a_2 - a_1) \\ &\quad + (a_1 - \tau a_0) + (1-\tau)|a_0| + |a_0|] \\ &= |z|^{n-k}|a_n z^{k+1} + (\sigma - 1)a_{n-k}| - |z|^n[|\rho| + \rho + a_n + (\sigma - 1)a_{n-k} \\ &\quad + 2|a_0| - \tau(|a_0| + a_0)] \qquad > 0 \end{aligned}$$

if

$$\left|z^{k+1}+\gamma_3\right|>\delta_3|z|^k,$$

where

$$\gamma_3=\frac{(\sigma-1)a_{n-k}}{a_n}$$

and

$$\delta_3=\frac{|\rho|+\rho+a_n+(\sigma-1)a_{n-k}+2|a_0|-\tau(|a_0|+a_0)}{|a_n|}.$$

This inequality holds if $|z|^{k+1}-|\gamma_3|>\delta_3|z|^k$.

Thus all the zeros of P(z) whose modulus is greater than 1 lie in the disk $|z|\le k_3$, where k_3 is the positive root of the equation

$$K^{k+1}-\delta_3K^k-|\gamma_3|=0.$$

But the zeros of P(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le k_3$, since k_3 can be easily shown to be greater than 1. That proves the first part of the theorem.

To prove the second part, if $a_{n-k}>a_{n-k+1}$, then $a_{n-k}>a_{n-k-1}$ and we have

$$\begin{aligned}F(z)=&-a_nz^{n+1}-\rho z^n+(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+.....\\&+(a_{n-k+1}-\sigma a_{n-k})z^{n-k+1}+(\sigma-1)a_{n-k}z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}\\&+......+(a_2-a_1)z^2+(a_1-\tau a_0)z+(\tau-1)a_0+a_0\end{aligned}$$

Therefore, for $|z|>1$,

$$\begin{aligned}|F(z)|\ge&\left|a_nz^{n+1}+(1-\sigma)a_{n-k}z^{n-k+1}\right|-\Big|-\rho z^n+(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+....\\&+(a_{n-k+1}-\sigma a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}+......\\&+(a_2-a_1)z^2+(a_1-\tau a_0)z+(\tau-1)a_0+a_0\Big|\\\ge&|z|^{n-k+1}\left|a_nz^k+(1-\sigma)a_{n-k}\right|-|z|^n[|\rho|+(\rho+a_n-a_{n-1})+\frac{a_{n-1}-a_{n-2}}{|z|}+......\\&+\frac{a_{n-k+1}-\sigma a_{n-k}}{|z|^{k-1}}+\frac{a_{n-k}-a_{n-k-1}}{|z|^k}+......+\frac{|a_2-a_1|}{|z|^{n-2}}\\&+\frac{a_1-\tau a_0}{|z|^{n-1}}+\frac{(1-\tau)|a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}]\\>&|z|^{n-k+1}\left|a_nz^k+(1-\sigma)a_{n-k}\right|-|z|^n[|\rho|+(\rho+a_n-a_{n-1})+(a_{n-1}-a_{n-2})+......\\&+(a_{n-k+1}-\sigma a_{n-k})+(a_{n-k}-a_{n-k-1})+......+(a_2-a_1)\\&+(a_1-\tau a_0)+(1-\tau)|a_0|+|a_0|]\\=&|z|^{n-k+1}\left|a_nz^k+(1-\sigma)a_{n-k}\right|-|z|^n[|\rho|+\rho+a_n+(1-\sigma)a_{n-k}\\&+2|a_0|-\tau(|a_0|+a_0)]\quad>0\end{aligned}$$

if

$$\left|z^k+\gamma_4\right|>\delta_4|z|^{k-1},$$

where

$$\gamma_4=\frac{(1-\sigma)a_{n-k}}{a_n}$$

and

$$\delta_4=\frac{|\rho|+\rho+a_n+(1-\sigma)a_{n-k}+2|a_0|-\tau(|a_0|+a_0)}{|a_n|}.$$

This inequality holds if $|z|^k-|\gamma_4|>\delta_4|z|^{k-1}$.

Thus all the zeros of P(z) whose modulus is greater than 1 lie in the disk $|z|\le k_4$, where k_4 is the positive root of the equation

$$K^k-\delta_4K^{k-1}-|\gamma_4|=0.$$

But the zeros of P(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le k_4$, since k_4 can be easily shown to be greater than 1. That proves the second part of the theorem and hence Theorem 1 is proved completely .

Proof of Theorem 2: Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......\\&\quad+(a_2-a_1)z^2+(a_1-a_0)z+a_0\\&=-a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+.....+(\alpha_2-\alpha_1)z^2\\&\quad+(\alpha_1-\alpha_0)z+\alpha_0+i\beta_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j.\end{aligned}$$

If $\alpha_{n-k-1}>\alpha_{n-k}$, then $\alpha_{n-k+1}>\alpha_{n-k}$ and we have

$$\begin{aligned}F(z)=-a_nz^{n+1}-\rho z^n&+(\rho+\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+.....(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}\\&+(\sigma\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}-(\sigma-1)\alpha_{n-k}z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......\\&+(\alpha_2-\alpha_1)z^2+(\alpha_1-\tau\alpha_0)z+(\tau-1)\alpha_0z+\alpha_0\quad.\\&+i\beta_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j.\end{aligned}$$

For $|z|>1$,

$$|F(z)|\ge\left|a_nz^{n+1}+(\sigma-1)\alpha_{n-k}z^{n-k}\right|-|z|^n[|\rho|+\rho+\alpha_n-\alpha_{n-1}+\frac{\alpha_{n-1}-\alpha_{n-2}}{|z|}+......$$

$$+\frac{\alpha_{n-k+1}-\alpha_{n-k}}{|z|^{k-1}}+\frac{\sigma\alpha_{n-k}-\alpha_{n-k-1}}{|z|^{k}}+\frac{\alpha_{n-k-1}-\alpha_{n-k-2}}{|z|^{k+1}}+......+\frac{\alpha_2-\alpha_1}{|z|^{n-2}}$$

$$+\frac{\alpha_1-\tau\alpha_0}{|z|^{n-1}}+\frac{(1-\tau)|\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^{n}}+\frac{|\beta_0|}{|z|^{n}}+\frac{\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)}{|z|^{n-j}}]$$

$$>\left|a_n z^{n+1}+(\sigma-1)\alpha_{n-k}z^{n-k}\right|-|z|^n[|\rho|+\rho+\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......$$

$$+\alpha_{n-k+1}-\alpha_{n-k}+\sigma\alpha_{n-k}-\alpha_{n-k-1}+\alpha_{n-k-1}-\alpha_{n-k-2}+......+\alpha_2-\alpha_1$$

$$+\alpha_1-\tau\alpha_0+(1-\tau)|\alpha_0|+|\alpha_0|+|\beta_n|+2\sum_{j=0}^{n-1}|\beta_j|]$$

$$=|z|^{n-k}\left|a_n z^{k+1}+(\sigma-1)\alpha_{n-k}\right|-|z|^n[|\rho|+\rho+\alpha_n+(\sigma-1)\alpha_{n-k}+2|\alpha_0|$$

$$-\tau(|\alpha_0|+\alpha_0)+|\beta_n|+2\sum_{j=0}^{n-1}|\beta_j|]$$

$$>0$$

if

$$\left|z^{k+1}+\gamma_7\right|>\delta_7|z|^k,$$

where

$$\gamma_7=\frac{(\sigma-1)\alpha_{n-k}}{a_n}$$

and

$$\delta_7=\frac{|\rho|+\rho+\alpha_n+(\sigma-1)\alpha_{n-k}+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+|\beta_n|+2\sum_{j=0}^{n-1}|\beta_j|}{a_n}.$$

This inequality holds if $|z|^{k+1}-|\gamma_7|>\delta_7|z|^k$.

Thus all the zeros of P(z) whose modulus is greater than 1 lie in the disk $|z|\le k_7$, where k_7 is the positive root of the equation

$$K^{k+1}-\delta_3K^k-|\gamma_3|=0.$$

It is easy to show that $k_7>1$, so that all those zeros of P(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le k_7$. Hence it follows that in this case all the zeros of P(z) lie in the disk $|z|\le k_7$,thereby proving the first part of the theorem.

For the second part, if $a_{n-k}>a_{n-k+1}$, then $a_{n-k}>a_{n-k-1}$ and we have

$$
\begin{aligned}
F(z) = & -a_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1}) z^n + (\alpha_{n-1} - \alpha_{n-2}) z^{n-1} + \dots\dots + (\alpha_{n-k+1} - \sigma\alpha_{n-k}) z^{n-k+1} \\
& - (1-\sigma)\alpha_{n-k} z^{n-k+1} + (\alpha_{n-k} - \alpha_{n-k-1}) z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2}) z^{n-k-1} + \dots\dots \\
& + (\alpha_2 - \alpha_1) z^2 + (\alpha_1 - \tau\alpha_0) z + (\tau - 1)\alpha_0 z + \alpha_0 \\
& + i\beta_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1}) z^j .
\end{aligned}
$$

For $|z| > 1$,

$$
\begin{aligned}
|F(z)| \geq & \left| a_n z^{n+1} + (1-\sigma)\alpha_{n-k} z^{n-k+1} \right| - |z|^n [|\rho| + \rho + \alpha_n - \alpha_{n-1} + \frac{\alpha_{n-1} - \alpha_{n-2}}{|z|} + \dots\dots \\
& + \frac{\alpha_{n-k+1} - \sigma\alpha_{n-k}}{|z|^{k-1}} + \frac{\alpha_{n-k} - \alpha_{n-k-1}}{|z|^k} + \frac{\alpha_{n-k-1} - \alpha_{n-k-2}}{|z|^{k+1}} + \dots\dots + \frac{\alpha_2 - \alpha_1}{|z|^{n-2}} \\
& + \frac{\alpha_1 - \tau\alpha_0}{|z|^{n-1}} + \frac{(1-\tau)|\alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + \frac{|\beta_0|}{|z|^n} + \frac{\sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|)}{|z|^{n-j}}] \\
> & \left| a_n z^{n+1} + (1-\sigma)\alpha_{n-k} z^{n-k+1} \right| - |z|^n [|\rho| + \rho + \alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \dots\dots \\
& + \alpha_{n-k+1} - \sigma\alpha_{n-k} + \alpha_{n-k} - \alpha_{n-k-1} + \alpha_{n-k-1} - \alpha_{n-k-2} + \dots\dots + \alpha_2 - \alpha_1 \\
& + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\alpha_0| + |\beta_n| + 2\sum_{j=0}^{n-1} |\beta_j|] \\
= & |z|^{n-k+1} \left| a_n z^k + (1-\sigma)\alpha_{n-k} \right| - |z|^n [|\rho| + \rho + \alpha_n + (1-\sigma)\alpha_{n-k} + 2|\alpha_0| \\
& - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1} |\beta_j|] \\
> & 0
\end{aligned}
$$

if

$$
\left| z^k + \gamma_8 \right| > \delta_8 |z|^{k-1},
$$

where

$$
\gamma_8 = \frac{(1-\sigma)\alpha_{n-k}}{a_n}
$$

and

$$
\delta_8 = \frac{|\rho| + \rho + \alpha_n + (1-\sigma)\alpha_{n-k} + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1} |\beta_j|}{a_n} .
$$

This inequality holds if $|z|^k - |\gamma_8| > \delta_8 |z|^{k-1}$.

Thus all the zeros of P(z) whose modulus is greater than 1 lie in the disk $|z| \leq k_8$, where k_8 is the positive root of the equation

$$
K^k - \delta_8 K^{k-1} - |\gamma_8| = 0 .
$$

It is easy to show that $k_8 > 1$, so that all those zeros of P(z) whose modulus is less than or equal to 1 are already contained in the disk $|z| \le k_8$. Hence it follows that in this case all the zeros of P(z) lie in the disk $|z| \le k_8$,thereby proving the second part of the theorem. That proves the theorem completely.

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