Deduction of Some Relations Connecting Bicomplex Analysis and Fluid Dynamics

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ABSTRACT

Bicomplex analysis is the most recent mathematical tool to develop the theory of complex analysis and its applications. In this paper we wish to establish some results connecting Bicomplex analysis with Fluid dynamics as a continuation of our earlier approach [15].

AMS Subject Classification (2010): 30D30, 30D35, 76A02

Keywords and phrases: Potential fluid flow, order (lower order), hyper order (hyper lower order), generalized order (generalized lower order), zero order (zero lower order), complex potential, composition, growth indicators, bicomplex number, bicomplex space, idempotent representation, factorization.

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1 INTRODUCTION .

In 1892, Corrado Segre [9] published a pioneer paper in which he introduced a generalization in the concept of Complex numbers, called Bicomplex numbers, Tricomplex numbers, etc. Thereafter, a numbers of renowned Mathematicians, namely, Michiji Futagawa [2], E. Hille [3], D. Riley [4], G. Baley Price [1] worked on the development of the subject.

Growth properties of complex functions regarding the value distribution theory of complex analysis are a broad area of research nowadays. Several properties of potential fluid flow have also been established regarding the application of complex analysis [11]. In this paper we establish some results connecting bicomplex analysis and fluid dynamics by introducing some new results.

2 DEFINITIONS AND NOTATIONS.

We denote **C_2** as a set of bicomplex numbers and **C_1** or usual **C** as a set of complex numbers. We have used some useful definitions and notations as mentioned below :

Definition 1. Bicomplex numbers. The bicomplex numbers are defined as $T = \{z_1 + i_2 z_2 / z_1, z_2 \in C(i_1)\}$, where the imaginary units i_1, i_2 follow the rules $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j$, say and $j^2=1$*etc.*

Another representation is: $T = \{w_0 + w_1 i_1 + w_2 i_2 + w_3 j / w_i \in R, i = 0,1,2,3\}$

Definition 2. Conjugate of a bicomplex number. Three types of conjugates can be defined of a bicomplex number $w = \{z_1 + i_2 z_2 / z_1, z_2 \in C(i_1)\}$ mentioned as follows:

a) $w_1' = (z_1 + i_2 z_2)' = \bar{z}_1 + i_2 \bar{z}_2$
b) $w_2' = (z_1 + i_2 z_2)' = z_1 - i_2 z_2$
c) $w_3' = (z_1 + i_2 z_2)' = \bar{z}_1 - i_2 \bar{z}_2$, where $\bar{z}_k = z_k'(C_1)$.

If $w = \{w_0 + w_1 i_1 + w_2 i_2 + w_3 j$ has the signature (+ + + +), then the conjugates have the signatures (+ − + −), (+ + − −) and (+ − − +) respectively.

Therefore, the composition of conjugates i.e, $\{w_0', w_1', w_2', w_3'\}$ forms a Klein – Group.

Definition 3. Idempotent representation of a bicomplex number. Every bicomplex number $(z_1 + i_2 z_2)$ has the following idempotent representation: $z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$, where $e_1 = \frac{1 + i_1 i_2}{2}, e_2 = \frac{1 - i_1 i_2}{2}$.

Definition 4. Bicomplex holomorphic functions. Let U be an open set of T and $w_0 \in U$. Then $f : U \subseteq T \to T$ is said to be T-differentiable at w0 with derivative equal to $f'(w_0) \in T$ if

$$\lim_{w \to w_0} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).$$

So, f is T-holomorphic in U if f is T-differentiable in U.

Definition 5. Bicomplex meromorphic functions. In complex plane, a function f is meromorphic in an open set U if and only if f is a quotient g/h of two functions g and h, holomorphic in U and h is not zero in U.

In bicomplex number, a function f is said to be bicomplex meromorphic in an open set $X \subset T$ if f is a quotient g/h ; g, h are holomorphic in X and h is not zero in X.

Definition 6. Idempotent representation of a bicomplex function. Let X_1, X_2 be open sets in $C(i_1)$ and $T \subset C(i_2)$. Then any bicomplex function $f(w) = f(z_1 + i_2 z_2) : X_1 \times_e X_2 \to T$ can be uniquely represented as follows:

$$f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2 \text{ for all } z_1 + i_2 z_2 \in X_1 \times X_2,$$

where $f_{e_1} : X_1 \to C(i_1)$ and $f_{e_2} : X_2 \to C(i_1)$ are two different complex functions.

Definition 7. Idempotent representation of a bicomplex holomorphic function. Let X_1, X_2 be open sets in $C(i_1)$ and $T \subset C(i_2)$. Then a bicomplex function $f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ for all $z_1 + i_2 z_2 \in X_1 \times_e X_2$, is said to be T-holomorphic if and only if $f_{e_1} : X_1 \to C(i_1)$ and $f_{e_2} : X_2 \to C(i_1)$ are holomorphic complex functions and $f'(z_1 + i_2 z_2) = f'_{e_1}(z_1 - i_1 z_2)e_1 + f'_{e_2}(z_1 + i_1 z_2)e_2$.

Definition 8. Idempotent representation of a bicomplex meromorphic function. Let X_1, X_2 be open sets in $C(i_1)$ and $T \subset C(i_2)$. Then a bicomplex function $f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ for all $z_1 + i_2 z_2 \in X_1 \times_e X_2$, is said to be meromorphic if and only if $f_{e_1} : X_1 \to C(i_1)$ and $f_{e_2} : X_2 \to C(i_1)$ are meromorphic complex functions.

Definition 9. Bicomplex transcendental meromorphic function. A function $f : T \to T$ is said to be a transcendental meromorphic function on T if and only if $f_{e_i} : C(i_1) \to C(i_1)$ are transcendental meromorphic functions for $i = 1,2$.

Definition 10. Factorization of a bicomplex meromorphic function. Let F be a bicomplex meromorphic function on T. Then f is said to have f and g as left and right factors respectively if F_{ei} has f_{ei} and g_{ei} as left and right factors for $i = 1,2$. Then we can write it as $F(w) = f(g(w))$.

Definition 11 [1]. Pole (Strong Pole) of a bicomplex function. Let $f : X \to T$ be a bicomplex meromorphic function on the open set $X \subset T$. We can say that $w = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \in X$ is a (strong) pole for the bicomplex meromorphic function $f(w) = f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ if $z_1 - i_1 z_2 \in P_1(X)$ and $z_1 + i_1 z_2 \in P_2(X)$ are poles for $f_{e_1} : P_1(X) \to C(i_1)$ and $f_{e_2} : P_2(X) \to C(i_1)$ respectively.

Remark 1. Poles of bicomplex meromorphic functions are not isolated singularities.

Proposition 1. Let $f : X \to T$ be a bicomplex meromorphic function on the open set $X \subset T$. If $w_0 \in X$ then w_0 is a pole of f, if and only if $\lim_{w \to w_0} |f(w)| = \infty$.

Definition 12. Order of a bicomplex function. The order ρ (F) of a bicomplex meromorphic function

$F(w) = F_{e_1}(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$ is defined as $\rho(F) = Max\{\rho_{F_{e_1}}, \rho_{F_{e_2}}\}$
where $\rho_{F_{e_i}} = \limsup_{r_i \to \infty} \frac{\log\log M_i(r_i, F_{e_i})}{\log r_i}$ for i = 1,2.

Remark 2. The lower order $\lambda(F)$ of a bicomplex meromorphic function is defined as
$\lambda(F) = Min\{\lambda(F_{e_1}), \lambda(F_{e_2})\}$. where $\lambda_{F_{e_i}} = \liminf_{r_i \to \infty} \frac{\log\log M_i(r_i, F_{e_i})}{\log r_i}$ for i = 1,2.

Remark 3. The hyper order $\bar{\rho}(F)$ (Hyper lower order $\bar{\lambda}(F)$) and the generalized order $\rho^{(k)}(F)$ (generalized lower order $\lambda^{(k)}(F)$) can also be defined in a similar way.

Definition 13. The type of F. The type $\sigma(F)$ of a bicomplex meromorphic function is defined as $\sigma(F) = Max\{\sigma(F_{e_1}), \sigma(F_{e_2})\}$ where $\sigma(F_{e_i}) = \limsup_{r_i \to \infty} \frac{\log M_i(r_i, F_{e_i})}{r_i^{\rho_{F_{ei}}}}$ and $0 < \rho_{F_{e_i}} < \infty$ for i =1,2.

Definition 14. Quantities $\rho^*(F) and \lambda^*(F)$: Let *F(w)* be an entire function order zero. Then $\rho^*(F) and \lambda^*(F)$ can be defined as $\rho^*(F) = Max\{\rho^*_{F_{e_1}}, \rho^*_{F_{e_2}}\}$ and $\lambda^*(F) = Min\{\lambda^*_{F_{e_1}}, \lambda^*_{F_{e_2}}\}$ where $\rho^*_{F_{e_i}} = \limsup_{r_i \to \infty} \frac{\log\log M_i(r_i, F_{e_i})}{\log\log r_i}$ and $\lambda^*_{F_{e_i}} = \liminf_{r_i \to \infty} \frac{\log\log M_i(r_i, F_{e_i})}{\log\log r_i}$ for i = 1,2.

Definition 15. Quantities $\rho^{**}(F)$ and $\lambda^{**}(F)$. Let F(w) be an entire function order zero. Then $\rho^{**}(F)$ and $\lambda^{**}(F)$. can be defined as $\rho^{**}(F) = Max\{\rho^{**}_{F_{e_1}}, \rho^{**}_{F_{e_2}}\}$ and $\lambda^{**}(F) = Min\{\lambda^{**}_{F_{e_1}}, \lambda^{**}_{F_{e_2}}\}$
where $\rho^{**}_{F_{e_i}} = \limsup_{r_i \to \infty} \frac{\log M_i(r_i, F_{e_i})}{\log r_i}$ and $\lambda^{**}_{F_{e_i}} = \liminf_{r_i \to \infty} \frac{\log M_i(r_i, F_{e_i})}{\log r_i}$ for i =1,2.

Definition 16 [13]. Factorization of F(w). Let F(w) be a bicomplex meromorphic function on $T \subset C_2$. Then F is said to have f and g as left and right factors respectively if F_{ei} has f_{ei} and g_{ei} as left and right factors respectively for i = 1,2, i.e., f_{ei} is meromorphic and g_{ei} is entire for i =1,2.

Definition 17. Complex potential fluid flow. If $f(z) = u(x, y) + iv(x, y) \in C_1$ be a complex function where $u(x, y) \in R^2$ and $v(x, y) \in R^2$ satisfy the Cauchy-Riemann equations, i.e, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and Laplace's equation, i.e, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, then *f(z)* can be termed as a complex potential fluid flow.

Definition 18. Bicomplex potential fluid flow. Similar to the complex potential fluid flow, if $f(w) = f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ be the idempotent composition of two complex functions, with $f_{e_1}(z_1 - i_1 z_2) = u(z_1, z_2) - i_1 v(z_1, z_2) \in C_1$ and $f_{e_2}(z_1 - i_1 z_2) = u(z_1, z_2) + i_1 v(z_1, z_2) \in C_1$ where *u(z1,z2)* and *v(z1,z2)* satisfy Cauchy-Riemann equations and Laplace's equation, i.e, $\frac{\partial u}{\partial z_1} = \frac{\partial v}{\partial z_2}$, $\frac{\partial u}{\partial z_2} = -\frac{\partial v}{\partial z_1}$ and $\frac{\partial^2 u}{\partial z_1^2} + \frac{\partial^2 u}{\partial z_2^2} = 0$, $\frac{\partial^2 v}{\partial z_1^2} + \frac{\partial^2 v}{\partial z_2^2} = 0$, therefore, $f_{e_1}(z_1 - i_1 z_2)$ and $f_{e_2}(z_1 + i_1 z_2)$ can be termed as complex potential fluid flows. So, *f(w)* can be termed as a composition of two different potential fluid flows f_{e_1} and f_{e_2}.

3 LEMMA.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [8][15]. If $f(z) = u(x, y) + iv(x, y)$ be complex potential fluid flow defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ is continuously differentiable in the region $\{y \geq 0\}$,

(ii) $f'(z)$ is parallel to the x - axis when y =0 and

(iii) $f'(z)$ is uniformly bounded in $\{y > 0\}$, then the order and lower order of $f(z)$ are zero.

Corollary 1. If $f(w) = f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ be an idempotent composition of two complex potential fluid flows satisfying the following properties:

(i) f_{e_1} and f_{e_2} are continuously differentiable in the region $\{y \geq 0\}$

(ii) f'_{e_1} and f'_{e_2} are parallel the x-axis when y = 0 and

(iii) f'_{e_1} and f'_{e_2} are uniformly bounded in $\{y > 0\}$, then the order and lower order of f(w) are zero.

Lemma 2 [14]. If f(z) and g(z) are any two entire functions, then for all sufficiently large values of r,

$$M\left(\frac{1}{8}M(\frac{r}{2}, g) - |g(0), f|\right) \leq M(r, f_o g) \leq M(M(r, g), f).$$

Lemma 3 [14]. If f be entire and g be a transcendental entire function of finite lower order, then for any $\delta > 0$,

$$M\left(r^{1+\delta}, f_o g\right) \geq M(M(r, g), f). \qquad (r \geq r_o)$$

Lemma 4 [13]. If F has f and g as left and right factors respectively, then we always have the following factorization:

F(w) = f (g (w)) .

4 THEOREMS.

In this section we present our main results of the paper.

Theorem 1. *If* $f(w) = f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ *be an idempotent composition of two complex potential fluid flows* f_{e1} and f_{e2} *satisfying the following properties:*

(i) f_{e1} and f_{e2} *are continuously differentiable in the region* $\{y \geq 0\}$

(ii) f'_{e1} and f'_{e2} *are parallel the x-axis when y = 0 and*

(iii) f'_{e1} and f'_{e2} *are uniformly bounded in* $\{y > 0\}$,

Then $\rho_f^* = 1$ *and* $\lambda_f^* = 1$.

Proof. From the definition of $\rho^{**}(f)$ and $\lambda^{**}(f)$ and using Definition 17, we have for arbitrary positive $\varepsilon_1, \varepsilon_2$ and all sufficiently large values of r_1, r_2,

$$\log M_1(r_1, f_{e_1}) \leq (\rho^{**}_{f_{e1}} + \varepsilon_1)\log r_1 \quad \text{and} \quad \log M_2(r_2, f_{e_2}) \leq (\rho^{**}_{f_{e2}} + \varepsilon_2)\log r_2$$

Therefore, $\log\log M_1(r_1, f_{e1}) \leq \log\log r_1 + O(1)$ and $\log\log M_2(r_2, f_{e2}) \leq \log\log r_2 + O(1)$

i.e. $$\frac{\log\log M_1(r_1, f_{e_1})}{\log\log r_1} \leq \frac{\log\log r_1 + O(1)}{\log\log r_1} \quad \text{and} \quad \frac{\log\log M_2(r_2, f_{e_2})}{\log\log r_2} \leq \frac{\log\log r_2 + O(1)}{\log\log r_2}$$

i.e. $$\limsup_{r_1 \leftarrow \infty} \frac{\log\log M_1(r_1, f_{e_1})}{\log\log r_1} \leq 1 \quad \text{and} \quad \limsup_{r_2 \leftarrow \infty} \frac{\log\log M_2(r_2, f_{e_2})}{\log\log r_2} \leq 1$$

i.e. $\rho^*_{f_{e1}} \leq 1$ and $\rho^*_{f_{e2}} \leq 1$ and therefore using Definition 14, we have $\rho_f^* \leq 1$. (1)

Similarly, proceeding as above and using Definition 14, we have $\lambda_f^* \leq 1$. (2)

Again, for arbitrary positive $\varepsilon_1, \varepsilon_2$ and all sufficiently large values of r_1, r_2 we have

$$\log M_1(r_1, f_{e_1}) \geq (\lambda^{**}_{f_{e1}} - \varepsilon_1)\log r_1 \quad \text{and} \quad \log M_2(r_2, f_{e2}) \geq (\lambda^{**}_{fe2} - \varepsilon_2)\log r_2.$$

Therefore, $\log\log M_1(r_1, f_{e_1}) \geq (\log\log r_1 + O(1)$ and $\log\log M_2(r_2, f_{e2}) \geq \log\log r_2 + O(1)$

i.e. $$\frac{\log\log M_1(r_1, f_{e_1})}{\log\log r_1} \geq \frac{\log\log r_1 + O(1)}{\log\log r_1}$$

and $$\frac{\log\log M_2(r_2, f_{e_2})}{\log\log r_2} \geq \frac{\log\log r_2 + O(1)}{\log\log r_2}$$

i.e. $$\limsup_{r_1 \leftarrow \infty} \frac{\log\log M_1(r_1, f_{e_1})}{\log\log r_1} \geq 1$$

and $$\limsup_{r_2 \leftarrow \infty} \frac{\log\log M_2(r_2, f_{e_2})}{\log\log r_2} \geq 1$$

i.e. $\rho^*_{f_{e1}} \geq 1$ and $\rho^*_{f_{e2}} \geq 1$ and therefore using Definition 14, we have $\rho^*_f \geq 1$. (3)

Similarly, proceeding as above and using Definition 14, we have $\lambda^*_f \geq 1$. (4)

From (1) and (3) we have $\rho^*_f = 1$ and from (2) and (4) we have $\lambda^*_f = 1$.

Hence the theorem follows.

Example 1. *Let* $f(w) = w = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \in C_2$ (bicomplex space) be a bicomplex potential fluid flow in $\mathbf{C_2}$.

Therefore, $f_{e_1} = z_1 - i_1 z_2 \in C_1(or, C)$ and $f_{e_2} = z_1 + i_1 z_2 \in C_1(or, C)$

Therefore, $\rho_{f_{e1}} = \limsup_{r_1 \to \infty} \frac{\log\log M_1(r_1, f_{e_1})}{\log r_1} = 0$ as $M_1(r_1, f_{e_1}) \leq |z_1 - i_1 z_2| \leq r_1 + r_2$

and $\rho_{f_{e2}} = \limsup_{r_2 \to \infty} \frac{\log\log M_2(r_2, f_{e_2})}{\log r_2} = 0$ as $M_2(r_2, f_{e_2}) \leq |z_1 + i_1 z_2| \leq r_1 + r_2$

Hence $\rho^*_f = 0$ and similarly $\lambda^*_f = 0$.

Now $\rho^*{}_{f_{e1}} = \limsup_{r_1 \to \infty} \frac{\log\log M_1(r_1, f_{e_1})}{\log\log r_1} = \limsup_{r_1 \to \infty} \frac{\log\log(r_1 + r_2)}{\log\log r_1} = 1$, r_2 is fixed.

And $\rho^*{}_{f_{e2}} = \limsup_{r_2 \to \infty} \frac{\log\log M_2(r_2, f_{e_2})}{\log\log r_2} = \limsup_{r_2 \to \infty} \frac{\log\log(r_1 + r_2)}{\log\log r_2} = 1$, r_1 is fixed.

Therefore, $\rho^*_f = 1$ and similarly $\lambda^*_f = 1$.

Similarly, we can also show that $\rho^{**}_f = 1$ and $\lambda^{**}_f = 1$.

Theorem 2. *Let f(w) and g(w) be any two bicomplex potential fluid flows satisfying the following properties:*

(i) f_{e_1}, f_{e_2} *are continuously differentiable in the region* $\{y \geq 0\}$

(ii) f'_{e_1}, f'_{e_2} *are parallel the x-axis when y = 0 and*

(iii) f'_{e_1}, f'_{e_2} *are uniformly bounded in* $\{y > 0\}$, *such that* $\rho_f = 0$ *and* $\lambda_g < \infty$.

Then $\rho_{f_o g} = \rho_g$.

Proof. Using Lemma 4 we can say that *F(w)* can be factorized to *f (g (w))*. Now, using Lemma 2 and Theorem 1 we have

$$\rho_{F_{e_1}} = \rho_{f_{e1}og_{e1}} = \limsup_{r_1\to\infty} \frac{\log\log M_1(r_1, f_{e_1}og_{e_1})}{\log r_1}$$

$$\leq \limsup_{r_1\to\infty} \frac{\log\log M(M(r_1, g_{e_1}), f_{e_1})}{\log\log M(r_1, g)} . \limsup_{r_1\to\infty} \frac{\log\log M(r_1, g_{e_1})}{\log r_1}$$

$$\leq \rho^*_{f_{e1}} . \rho_{g_{e1}} = 1.\rho_{g_{e1}} = \rho_{g_{e1}}$$

Similarly $\rho_{F_{e2}} \leq \rho_{g_{e2}}$.

Therefore $\rho_F = \rho_{f_o g} \leq Max\{\rho_{g_{e1}}, \rho_{g_{e2}}\} = \rho_g$, i.e. $\rho_F \leq \rho_g$. (5)

Now using Lemma 3 and Theorem 1, we have

$$\rho_{F_{e_1}} = \rho_{f_{e1}og_{e1}} = \limsup_{r_1\to\infty} \frac{\log\log M_1(r_1^{1+\delta}, f_{e1}og_{e_1})}{\log r_1^{1+\delta}}$$

$$\geq \liminf_{r_1\to\infty} \frac{\log\log M(M(r_1, g_{e_1}), f_{e_1})}{\log\log M(r_1, g_{e_1})} . \limsup_{r_1\to\infty} \frac{\log\log M(r_1, g_{e_1})}{\log r_1}$$

$$\geq \lambda^*_{f_{e1}} . \rho_{g_{e1}} = 1.\rho_{g_{e1}} = \rho_{g_{e1}}.$$

Similarly $\rho_{F_{e2}} \geq \rho_{g_{e2}}$.

Therefore $\rho_F = \rho_{f_o g} \geq Max\{\rho_{g_{e1}}, \rho_{g_{e2}}\} = \rho_g$, i.e. $\rho_F \geq \rho_g$. (6)

Therefore from (5) and (6), the result follows.

Theorem 3. *Let f(w) and g(w) be any two bicomplex potential fluid flows satisfying the following properties:*

(i) *f(w) is entire and g(w) is transcendental such that* $\rho_{fog} = 0$ *and* $\lambda_g < \infty$.

Then $\rho_f^{**} \lambda_g^{**} \leq \rho_{fog}^{**} \leq \rho_f^{**} \rho_g^{**}$.

Proof. Using Lemma 3 we have

$$\rho^{**}_{F_{e_1}} = \rho^{**}{}_{f_{e1} o g_{e1}} = \limsup_{r_1 \to \infty} \frac{\log M_1(r_1^{1+\delta_1}, f_{e_1} o g_{e_1})}{\log r_1^{1+\delta_1}}$$

$$\geq \limsup_{r_1 \to \infty} \frac{\log M_1(M_1(r_1, g_{e_1}), f_{e_1})}{\log M_1(r_1, g)} . \liminf_{r_1 \to \infty} \frac{\log M_1(r_1, g_{e_1})}{\log r_1}$$

$$= \rho^{**}_{f_{e1}} \lambda^{**}_{g_{e1}}.$$

Similarly, $\rho^{**}_{F_{e_2}} = \rho^{**}_{f_{e_2} o g_{e_2}} \geq \rho^{**}_{f_{e_2}} . \lambda^{**}_{g_{e_2}}$.

Therefore $\rho^{**}_{fog} = Max\{\rho^{**}_{f_{e_1} o g_{e1}}, \rho^{**}_{f_{e_2} o g_{e_2}}\} \geq \rho^{**}_f \lambda^{**}_g$. (7)

Again, by using Lemma 2 we have

$$\rho^{**}_{F_{e_1}} = \rho^{**}{}_{f_{e1} o g_{e1}} = \limsup_{r_1 \to \infty} \frac{\log M_1(r_1^{1+\delta_1}, f_{e_1} o g_{e_1})}{\log r_1^{1+\delta_1}}$$

$$\leq \limsup_{r_1 \to \infty} \frac{\log M_1(M_1(r_1, g_{e_1}), f_{e_1})}{\log M_1(r_1, g)} . \limsup_{r_1 \to \infty} \frac{\log M_1(r_1, g_{e_1})}{\log r_1}$$

$$= \rho^{**}_{f_{e1}} \rho^{**}_{g_{e1}}.$$

Similarly, $\rho^{**}_{F_{e_2}} = \rho^{**}_{f_{e_2} o g_{e_2}} \leq \rho^{**}_{f_{e_2}} . \rho^{**}_{g_{e_2}}$.

Therefore $\rho^{**}_{fog} = Max\{\rho^{**}_{f_{e_1} o g_{e_1}}, \rho^{**}_{f_{e_2} o g_{e_2}}\} \leq \rho^{**}_f \rho^{**}_g$. (8)

Therefore, from (7) and (8) the result follows.

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